

Altruistic defense traits in structured populations

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Abstract

We propose a model for the frequency of an altruistic defense trait. More precisely, we consider Lotka-Volterra-type models involving a host/prey population consisting of two types and a parasite/predator population where one type of host individuals (modeling carriers of a defense trait) is more effective in defending against the parasite but has a weak reproductive disadvantage. Under certain assumptions we prove that the relative frequency of these altruistic individuals in the total host population converges to spatially structured Wright-Fisher diffusions with frequency-dependent migration rates. For the many-demes limit (mean-field approximation) hereof, we show that the defense trait goes to fixation/extinction if and only if the selective disadvantage is smaller/larger than an explicit function of the ecological model parameters.

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1 Introduction

Altruism refers to the behavior of an individual that decreases the reproductive success of the actor while increasing the reproductive success of one or more recipients. In most natural systems, non-altruistic individuals benefit from altruistic individuals without suffering from the fitness disadvantage and, thus, have a direct reproductive advantage. So how can genetically inherited selfless behavior be explained by natural selection? This problem has bothered biologists since Charles Darwin who reflected the puzzle of sterile social insects such as the worker castes of ants in his famous book “The Origin of Species” [5].

In the biology and game theory literature there exist several explanations for the emergence of altruism (also referred to as cooperation in game theory). The central idea behind kin selection is that helping direct relatives benefits the reproductive success of the altruists’ genes. This idea is formalized in Hamilton’s rule which states that traits increase in frequency if $R \cdot B > C$ where R is the genetic relatedness of the recipient and the actor, B is the additional reproductive benefit gained by the recipient, and C is the reproductive cost to the actor; see Hamilton [13]. Relatedness is frequently defined as the probability of sharing the same allele by descent, e.g., $1/2$ for two sisters and $1/8$ for two cousins. However, general applicability of Hamilton’s rule is controversial; e.g., the paper by Nowak et al. [27] provoked a strong response including a rebuttal from 137 researchers [1]. Another explanation for the emergence of altruistic behavior is the intensively debated theory of group selection; see, e.g., Wade [40] and Queller [30]. The central idea is that groups of cooperators grow faster and, therefore, split earlier or into more groups than groups of defectors; see, e.g., the haystack model of Maynard Smith [23] or Traulsen and Nowak [37]. The importance of group selection (or more generally multilevel selection) in evolution remains controversial; cf., e.g., Maynard Smith [24], Goodnight and Stevens [11], Goodnight and Wade [12], Traulsen [36], and Gardner [9]. Further game theoretic explanations for the emergence of cooperative behavior include direct reciprocity with the repeated Prisoner’s Dilemma as a prominent example, indirect reciprocity, and network reciprocity. Reciprocity has also been observed in a number of animal taxa; see, e.g., Bshary and Grutter [3] or McGregor [25]. For comparative reviews of the above mentioned explanations, from different perspectives, see Nowak [26] and West et al. [41]. Moreover a number of recent papers propose spatially distributed predator-prey (or host-parasite) models and study these models via computer simulations; see, e.g., Commins et al. [4], Rand et al. [31], Haraguchi and Sasaki [14], Rauch et al. [32, 33], Goodnight et al. [10], Best et al. [2], and the references therein. In these models (except for Commins et al. [4]), points in a lattice change between the states “susceptible”, “infected”, and “unoccupied” according to probabilities that depend on the states of the neighboring lattice points.

In this article we focus on the important scenario of defense against a parasite or predator. Examples of such a scenario include self-sacrificial colony defense in social insects (see Shorter and Rueppell [35] for a review), suicidal defense of bacteria against pathogen infection (see Fukuyo et al. [8]), and slave rebellion in ants (see Pamminger et al. [28]). Clearly, close relatives of altruists are likely to live in the immediate vicinity and benefit from the altruists which increases the inclusive fitness of defense traits. However, Hamilton’s rule is difficult to apply if the relative frequency of related recipients is unknown. The theory of group selection contributes the qualitative explanation that demes with many altruists have a larger carrying capacity and, thereby, support more successful emigration events. However, it is difficult to calculate the selective advantage of a deme without knowing the local relative frequencies of altruists. So to get a quantitative model for altruistic defense against parasites, we will derive as our main contribution, Theorem 1.3 below, the dynamics of the local frequencies of altruists. In our model we only incorporate kin selection or group selection implicitly. In particular, we do not assume that altruists specifically favor close relatives or that competition among groups occurs within a few generations as in most of the traditional models on group selection; cf., e.g., Maynard Smith [23] and van Valen [38]. Instead, we begin with standard spatial host-parasite models (or, equivalently, with spatial predator-prey models) with a host population consisting of two types and a parasite population. One type of host individuals behaves altruistically in the sense that it has a reduced growth rate but also contributes less to the growth of the parasite population. So we do not incorporate the defense mechanism itself in our model but only its effect of reducing the per capita growth rate of the parasite population.

1.1 Main results

We begin with a stochastic extension of the classical and long-established Lotka-Volterra model (see Lotka [22] and Volterra [39]) which can be obtained as an approximation of discrete Markov chains such as renormalized

two-types birth and death processes in the case of large populations. To formulate these stochastic extensions, we consider the following setting (see Section 1.2 for notational conventions used throughout this article). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let \mathcal{D} be an at most countable set (the set of demes) and let $m \in [0, \infty)^{\mathcal{D} \times \mathcal{D}}$ satisfy for every $i \in \mathcal{D}$ that $\sum_{k \in \mathcal{D}} m(k, i) = \sum_{k \in \mathcal{D}} m(i, k) = 1$. We refer to m as the *migration matrix* or matrix of migration rates. Let $\lambda, K, \delta, \nu, \gamma, \eta, \rho \in (0, \infty)$ satisfy $\rho < \eta$. For every $N \in \mathbb{N}$, let $\kappa_H^N, \kappa_P^N, \alpha^N, \beta_H^N, \beta_P^N, \iota_H^N, \iota_P^N \in [0, \infty)$, let $W^{A,N}(i), W^{C,N}(i), W^{P,N}(i): [0, \infty) \times \Omega \rightarrow \mathbb{R}$, $i \in \mathcal{D}$, be independent Brownian motions with continuous sample paths, let $A^N, C^N, P^N: [0, \infty) \times \mathcal{D} \times \Omega \rightarrow [0, \infty)$ be adapted processes with continuous sample paths that for all $t \in [0, \infty)$ and all $i \in \mathcal{D}$ satisfy \mathbb{P} -a.s.

$$\begin{aligned}
A_t^N(i) &= A_0^N(i) + \int_0^t \kappa_H^N \sum_{j \in \mathcal{D}} m(i, j) (A_s^N(j) - A_s^N(i)) + A_s^N(i) \left[\lambda \left(1 - \frac{A_s^N(i) + C_s^N(i)}{K} \right) - \delta P_s^N(i) - \alpha^N \right] ds \\
&\quad + \int_0^t \iota_H^N \frac{A_s^N(i)}{A_s^N(i) + C_s^N(i)} ds + \int_0^t \sqrt{\beta_H^N A_s^N(i)} dW_s^{A,N}(i), \\
C_t^N(i) &= C_0^N(i) + \int_0^t \kappa_H^N \sum_{j \in \mathcal{D}} m(i, j) (C_s^N(j) - C_s^N(i)) + C_s^N(i) \left[\lambda \left(1 - \frac{A_s^N(i) + C_s^N(i)}{K} \right) - \delta P_s^N(i) \right] ds \\
&\quad + \int_0^t \iota_H^N \frac{C_s^N(i)}{A_s^N(i) + C_s^N(i)} ds + \int_0^t \sqrt{\beta_H^N C_s^N(i)} dW_s^{C,N}(i), \\
P_t^N(i) &= P_0^N(i) + \int_0^t \kappa_P^N \sum_{j \in \mathcal{D}} m(i, j) (P_s^N(j) - P_s^N(i)) ds \\
&\quad + \int_0^t P_s^N(i) [-\nu - \gamma P_s^N(i) + \eta C_s^N(i) + (\eta - \rho) A_s^N(i)] + \iota_P^N ds + \int_0^t \sqrt{\beta_P^N P_s^N(i)} dW_s^{P,N}(i),
\end{aligned} \tag{1}$$

let $H^N: [0, \infty) \times \mathcal{D} \times \Omega \rightarrow [0, \infty)$ satisfy $H^N = A^N + C^N$, and let $F^N: [0, \infty) \times \mathcal{D} \times \Omega \rightarrow [0, 1]$ satisfy $F^N = \frac{A^N}{H^N}$. For every $N \in \mathbb{N}$, the process H^N describes the host (or prey) populations, A^N describes the altruists (or cooperators), C^N describes the cheaters (or defectors), and P^N describes the parasite (or predator) populations, each measured in units of N individuals. Existence of solutions to (1), which we assume here, can be established in suitable Liggett-Spitzer spaces if \mathcal{D} is an Abelian group and if m is translation invariant and irreducible; cf. Proposition 2.1 in [17].

The central goal of this article is to prove convergence of the sequence $((\frac{A_t^N}{A_t^N + C_t^N})_{t \in [0, \infty)})_{N \in \mathbb{N}}$ and to derive the diffusion equation which the limit solves. In other words, we will derive an analog of the Kimura stepping stone model (i.e., spatially structured Wright-Fisher diffusions) for altruistic defense against parasites. Since A^N, C^N, P^N are measured in units of N individuals and the stochastic fluctuations scale with \sqrt{N} as $N \rightarrow \infty$, we need to assume that β_H^N and β_P^N are of order $\frac{1}{\sqrt{N}}$ for large $N \in \mathbb{N}$. To get a nontrivial diffusion approximation we additionally assume – as is usual in the derivation of the Kimura stepping stone model – slow migration and weak selection in the sense that the sequences $(N\kappa_H^N)_{N \in \mathbb{N}}$, $(N\kappa_P^N)_{N \in \mathbb{N}}$ and $(N\alpha^N)_{N \in \mathbb{N}}$ converge. Thus the relative frequency of altruists $\frac{A^N}{A^N + C^N}$ evolves on the time scale of order N as $N \rightarrow \infty$.

In the special case that for some $N \in \mathbb{N}$ it holds that $\kappa_H^N = \kappa_P^N = \iota_H^N = \iota_P^N = \alpha^N = \beta_H^N = \beta_P^N = A_0^N = 0$, then $A^N \equiv 0$ and $(H^N(i), P^N(i))$, $i \in \mathcal{D}$, satisfy classical Lotka-Volterra equations. It is well known that if $K\eta > \nu$, then the solutions of these equations converge to the nontrivial equilibrium $(\frac{K\delta\nu}{\lambda\gamma + K\delta\eta}, \frac{\lambda K\eta - \lambda\nu}{\lambda\gamma + \delta K\eta}) \in (0, \infty)^2$ in each deme. Since we assume that $\kappa_H^N, \kappa_P^N, \alpha^N, \iota_H^N, \iota_P^N, \beta_H^N, \beta_P^N$ are of order $o(1)$ as $N \rightarrow \infty$ and since the altruist frequencies evolve slowly, for every $i \in \mathcal{D}$, the processes $(H^N(i), P^N(i))$ should asymptotically be close to the equilibrium of the classical Lotka-Volterra equations with η being replaced by $\eta - \rho F^N(i)$ as $N \rightarrow \infty$. More precisely, we will prove in Theorem 1.2 below under further assumptions that if the local frequency of altruists is $q \in [0, 1]$, then the equilibrium state for hosts and parasites should be $(h_\infty(q), p_\infty(q))$ where the functions h_∞ and p_∞ are defined by

$$\begin{aligned}
[0, 1] \ni x \mapsto h_\infty(x) &:= \frac{K(\delta\nu + \gamma\lambda)}{\lambda\gamma + \delta K(\eta - \rho x)} = \frac{1}{b(a-x)} \in (0, \infty) \\
[0, 1] \ni x \mapsto p_\infty(x) &:= \frac{\lambda K(\eta - \rho x) - \lambda\nu}{\lambda\gamma + \delta K(\eta - \rho x)} = \frac{\lambda}{\delta} \left(1 - \frac{1}{Kb(a-x)} \right) \in (0, \infty)
\end{aligned} \tag{2}$$

and where $a := \frac{\lambda\gamma + \delta K\eta}{\delta K\rho}$ and $b := \frac{\delta\rho}{\delta\nu + \lambda\gamma}$. For these functions to be well defined we will assume that $Kb(a-1) > 1$

or, equivalently, that $K(\eta - \rho) > \nu$.

The above heuristic is incorrect if all populations go extinct by chance due to stochasticity in the offspring distributions. To avoid this difficulty we will assume that there is sufficient immigration of hosts ($2\iota_H^N \geq \beta_H^N$) and parasites ($2\iota_P^N \geq \beta_P^N$) in order that both host populations and parasite populations cannot go extinct; see Lemmas 2.2 and 2.3, respectively. However, note that both altruists and cheaters can locally die out. For our proof, which is based on the Lyapunov function (67), we additionally require further restrictions on the parameters and on (inverse) moments of the initial configuration.

Assumption 1.1. *In the setting of the first paragraph of Section 1.1 it holds that $\lambda > \nu$, $\eta - \rho > \frac{\lambda}{K}$, $\gamma \geq 2\delta$, for all $N \in \mathbb{N}$ it holds that $\alpha^N + \kappa_H^N \leq \frac{\lambda}{4}$, $\iota_P^N \leq \frac{\lambda(\nu+\lambda)}{8\delta}$, $\kappa_P^N + \kappa_H^N + \alpha^N \leq \frac{\lambda-\nu}{2}$, $\iota_H^N \geq \frac{4\delta\kappa_P^N}{3(\nu+\lambda)} + \frac{3}{2}\beta_H^N$, $\iota_P^N \geq \beta_P^N$, and there exist $\sigma = (\sigma_i)_{i \in \mathcal{D}} \in (0, \infty)^{\mathcal{D}}$ and $c \in (0, \infty)$ such that $\sum_{i \in \mathcal{D}} \sigma_i < \infty$, such that for every $j \in \mathcal{D}$ it holds that*

$$\sum_{i \in \mathcal{D}} \sigma_i m(i, j) \leq c\sigma_j, \quad (3)$$

and such that $\sup_{N \in \mathbb{N}} \mathbb{E} \left[\left\| \left(H_0^N + P_0^N \right)^4 + \frac{1}{(H_0^N)^2} + \frac{P_0^N}{(H_0^N)^2} + \frac{1}{P_0^N} + \frac{1}{P_0^N H_0^N} \right\|_{\sigma} \right] < \infty$.

The following theorem, which appears to be new even for non-spatial stochastic Lotka-Volterra stochastic differential equations (SDEs), proves for every $t \in [0, \infty)$ that the $L^2([0, t] \times l_{\sigma}^1 \times \Omega; \mathbb{R})$ -distance between $(H_{\cdot N}^N, P_{\cdot N}^N)$ and $(h_{\infty}(F_{\cdot N}^N), p_{\infty}(F_{\cdot N}^N))$ converges to 0 as $N \rightarrow \infty$ at least with rate $\frac{1}{2}$. Theorem 1.2 follows immediately from Theorem 2.8 below together with a time substitution.

Theorem 1.2. *Assume the setting of the first paragraph of Section 1.1, let Assumption 1.1 hold, assume that $\sup_{N \in \mathbb{N}} (N \max\{\kappa_H^N, \kappa_P^N, \alpha^N, \iota_H^N, \iota_P^N, \beta_H^N, \beta_P^N\}) < \infty$ and let h_{∞} and p_{∞} be given by (2). Then we get for all sets $\hat{\mathcal{D}} \subseteq \mathcal{D}$ and all $t \in [0, \infty)$ that*

$$\sup_{N \in \mathbb{N}} N \int_0^t \mathbb{E} \left[\sum_{i \in \hat{\mathcal{D}}} \sigma_i \left(H_{uN}^N(i) - h_{\infty}(F_{uN}^N(i)) \right)^2 + \sum_{i \in \hat{\mathcal{D}}} \sigma_i \left(P_{uN}^N(i) - p_{\infty}(F_{uN}^N(i)) \right)^2 \right] du < \infty. \quad (4)$$

Knowing the asymptotic behavior of the host populations, we can formally replace the $(H^N)_{N \in \mathbb{N}}$ in the diffusion equation (13) of the altruist frequencies and, thereby, we arrive at the diffusion equation which the limit of altruist frequencies solves. Our main result, Theorem 1.3, then proves that the altruist frequencies converge to the solution of the diffusion equation (5). The proof of Theorem 1.3 is deferred to Section 2.4.2 below and is based on a general stochastic averaging result in Kurtz [21].

Theorem 1.3. *Assume the setting of the first paragraph of Section 1.1, let Assumption 1.1 hold, assume that $\sum_{i \in \mathcal{D}} \sup_{N \in \mathbb{N}} \sigma_i \mathbb{E} \left[H_0^N(i) \right] < \infty$, that $\sup_{N \in \mathbb{N}} (N \max\{\kappa_P^N, \iota_H^N, \iota_P^N, \beta_P^N\}) < \infty$, that there exist $\kappa, \alpha, \beta \in [0, \infty)$ such that $\lim_{N \rightarrow \infty} \kappa_H^N N = \kappa$, $\lim_{N \rightarrow \infty} \alpha^N N = \alpha$ and $\lim_{N \rightarrow \infty} \beta_H^N N b = \beta$ and assume that $F_0^N \Rightarrow X_0$ as $N \rightarrow \infty$ in l_{σ}^1 . Then the SDE*

$$\begin{aligned} dX_t(i) = & \kappa \sum_{j \in \mathcal{D}} m(i, j) \frac{a - X_t(i)}{a - X_t(j)} \left(X_t(j) - X_t(i) \right) dt - \alpha X_t(i) (1 - X_t(i)) dt \\ & + \sqrt{\beta(a - X_t(i)) X_t(i) (1 - X_t(i))} dW_t(i), \quad t \in (0, \infty), i \in \mathcal{D} \end{aligned} \quad (5)$$

(where $\{W(i) : i \in \mathcal{D}\}$ are independent standard Brownian motions) has a unique strong solution and

$$(F_{tN}^N)_{t \in [0, \infty)} \Rightarrow (X_t)_{t \in [0, \infty)} \quad (6)$$

as $N \rightarrow \infty$ in $C([0, \infty), l_{\sigma}^1)$.

An important problem is to derive conditions under which altruists persist, that is, to derive conditions on the parameters of the SDE (5) under which the process goes to fixation. Here we simplify this problem and consider the many-demes-limit (also denoted as mean-field approximation) of the SDE (5). More precisely, for every $D \in \mathbb{N}$, let $X^D : [0, \infty) \times \{1, \dots, D\} \times \Omega \rightarrow [0, 1]$ be the solution of the SDE (5) with \mathcal{D} replaced by $\{1, \dots, D\}$ and

with m replaced by $(\frac{1}{D} \mathbb{1}_{i=j})_{i,j \in \{1, \dots, D\}}$. We will show in Proposition 3.1 together with Lemma 3.2 below that if, for every $D \in \mathbb{N}$, $(X_0^D(i))_{i \in \{1, \dots, D\}}$ are exchangeable $[0, 1]$ -valued random variables, if $\sup_{D \in \mathbb{N}} \mathbb{E}[(X_0^D(1))^2] < \infty$, if $Z: [0, \infty) \times \Omega \rightarrow [0, 1]$ is the solution of the SDE (8) below with respect to the Brownian motion $W(1)$ and if $\sup_{D \in \mathbb{N}} \sqrt{D} \mathbb{E}[|X_0^D(i) - Z_0(i)|] < \infty$, then for all $t \in [0, \infty)$ it holds that

$$\sup_{D \in \mathbb{N}} \sqrt{D} \mathbb{E}[|X_t^D(1) - Z_t|] < \infty. \quad (7)$$

Thus the solution of the SDE (8) is the many-demes limit of the SDE (5). For this many-demes limit we derive a simple necessary and sufficient condition ($\alpha < \beta$) under which the altruistic defense trait goes to fixation when starting with a positive frequency. The proof of Theorem 1.4 is deferred to Section 4.3.

Theorem 1.4. *Let $\alpha, \beta, \kappa \in (0, \infty)$, let $a \in (1, \infty)$, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0, \infty)})$ be a filtered probability space, let $W: [0, \infty) \times \Omega \rightarrow \mathbb{R}$ be a standard $(\mathcal{F}_t)_{t \in [0, \infty)}$ Brownian motion with continuous sample paths, and let $Z_0: \Omega \rightarrow [0, 1]$ be an $\mathcal{F}_0/\mathcal{B}([0, 1])$ -measurable mapping. Then the SDE*

$$dZ_t = \kappa(a - Z_t) \left((a - Z_t) \mathbb{E} \left[\frac{1}{a - Z_t} \right] - 1 \right) dt - \alpha Z_t (1 - Z_t) dt + \sqrt{\beta(a - Z_t)Z_t(1 - Z_t)} dW_t \quad (8)$$

has a unique solution. Furthermore, if $\mathbb{E}[Z_0] = 1$, then $\mathbb{P}[Z_t = 1 \text{ for all } t \in [0, \infty)] = 1$, if $\mathbb{E}[Z_0] = 0$, then $\mathbb{P}[Z_t = 0 \text{ for all } t \in [0, \infty)] = 1$ and if $\mathbb{E}[Z_0] \in (0, 1)$, then

$$\begin{aligned} \lim_{t \rightarrow \infty} \mathbb{E}[|Z_t - 0|] &= 0, \text{ if } \alpha > \beta, \\ \lim_{t \rightarrow \infty} \mathbb{E}[|Z_t - 1|] &= 0, \text{ if } \alpha < \beta, \\ Z_t &\xrightarrow{t \rightarrow \infty} \int m(z) dz, \text{ if } \alpha = \beta, \end{aligned} \quad (9)$$

where $m(z) = \frac{1}{c} z^{\frac{2\kappa}{\beta}(a\theta-1)-1} (1-z)^{\frac{2\kappa}{\beta}(1-\theta(a-1))-1} (a-z)^{\frac{2\alpha}{\beta}-1}$ for $z \in (0, 1)$, where $c \in (0, \infty)$ is a normalizing constant and where $\theta = \mathbb{E}[\frac{1}{a-Z_0}]$.

Informally speaking, Theorem 1.4 asserts that an altruistic defense allele persists in an infinite dimensional space if $\alpha < \beta$ and if the mean frequency of altruists over all demes is positive. This does not imply that a new mutation resulting in altruistic defense behavior can establish itself on one island or even in the total population. Our final result partially closes this gap and considers a process which could be the limit $\lim_{D \rightarrow \infty} \sum_{i=1}^D X^D(i)$ if for all $D \in \mathbb{N}$ and $i \in \{1, \dots, D\}$ it holds that $X_0^D(i) = Y_0 \mathbb{1}_{i=1}$ for some $[0, 1]$ -valued random variable; cf. Hutzenthaler [15], [16]. For this limiting process, Proposition 5.1 below shows in the case $\mathbb{P}[Y_0 > 0] = 1$ that the process converges to 0 in probability as time goes to infinity if and only if $\alpha \geq \beta$. Informally speaking, Proposition 5.1 asserts that an altruistic defense allele has a positive invasion probability in an infinite dimensional space if and only if $\alpha < \beta$.

1.2 Notation

Throughout this article, we will use the following notation. We define $[0, \infty] := [0, \infty) \cup \{\infty\}$. We will use the conventions that $0^0 = 1$, $0 \cdot \infty = 0$, and that for any $x \in (0, \infty)$ we have that $\frac{x}{\infty} = 0$ and $\frac{\infty}{0} = \infty$. For all $x, y \in \mathbb{R}$ we define $x^+ := \max\{x, 0\}$, $\text{sgn}(x) := \mathbb{1}_{x>0} - \mathbb{1}_{x<0}$, and $x \wedge y := \min\{x, y\}$. We define $\sup(\emptyset) := -\infty$ and $\inf(\emptyset) := \infty$. For a topological space (E, \mathcal{E}) we denote by $\mathcal{B}(E)$ the Borel sigma-algebra of (E, \mathcal{E}) . Moreover we agree on the convention that zero times an undefined expression is set to zero. For every countable set \mathcal{D} and every $\sigma = (\sigma_i)_{i \in \mathcal{D}} \in (0, \infty)^{\mathcal{D}}$ define a function $\|\cdot\|_{\sigma}: \mathbb{R}^{\mathcal{D}} \rightarrow [0, \infty]$ by $\mathbb{R}^{\mathcal{D}} \ni z = (z_i)_{i \in \mathcal{D}} \mapsto \|z\|_{\sigma} := \sum_{i \in \mathcal{D}} \sigma_i |z_i|$ and define $l_{\sigma}^1 := \{z \in \mathbb{R}^{\mathcal{D}}: \|z\|_{\sigma} < \infty\}$.

2 Convergence of the relative frequency of altruists

2.1 Setting

Assume the setting of the first paragraph of Section 1.1. Define $\bar{\kappa}_H := \sup_{N \in \mathbb{N}} \kappa_H^N$, $\bar{\kappa}_P := \sup_{N \in \mathbb{N}} \kappa_P^N$, $\bar{\beta}_H := \sup_{N \in \mathbb{N}} \beta_H^N$, $\bar{\beta}_P := \sup_{N \in \mathbb{N}} \beta_P^N$, $\bar{\iota}_H := \sup_{N \in \mathbb{N}} \iota_H^N$, $\bar{\iota}_P := \sup_{N \in \mathbb{N}} \iota_P^N$, and $\beta_H := \lim_{N \rightarrow \infty} \beta_H^N$. For all $z =$

$(z_i)_{i \in \mathcal{D}} \in (0, \infty)^{\mathcal{D}}$ and $p \in \mathbb{R}$ let $z^p = (z_i^p)_{i \in \mathcal{D}}$. Furthermore, let $\mathbf{1} := (1)_{i \in \mathcal{D}} \in l_\sigma^1$. Define $E_1 := [0, 1]^{\mathcal{D}}$ and $E_2 := l_\sigma^1 \cap [0, \infty)^{\mathcal{D}}$. For all $i \in \mathcal{D}$ and all $N \in \mathbb{N}$ let $W^{H,N}(i): [0, \infty) \times \Omega \rightarrow \mathbb{R}$ and $W^{F,N}(i): [0, \infty) \times \Omega \rightarrow \mathbb{R}$ be stochastic processes with continuous sample paths such that for every $t \in [0, \infty)$ it holds \mathbb{P} -a.s. that

$$dW_t^{H,N}(i) = \frac{\sqrt{A_t^N(i)} dW_t^{A,N}(i) + \sqrt{C_t^N(i)} dW_t^{C,N}(i)}{\sqrt{H_t^N(i)}} \quad (10)$$

and

$$dW_t^{F,N}(i) = \frac{\sqrt{C_t^N(i)} dW_t^{A,N}(i) - \sqrt{A_t^N(i)} dW_t^{C,N}(i)}{\sqrt{H_t^N(i)}}, \quad (11)$$

respectively, with $W_0^{H,N}(i) = W_0^{F,N}(i) = 0$.

2.2 Preliminaries

Assume the setting of Section 2.1. In this section we collect some first results that are used in the proofs of the statements in subsequent sections.

Lemma 2.1. *Assume the setting of Section 2.1. Then $W^{H,N}(i)$ and $W^{F,N}(i)$, $N \in \mathbb{N}$, $i \in \mathcal{D}$, are independent Brownian motions and for all $t \in [0, \infty)$, all $i \in \mathcal{D}$, and all $N \in \mathbb{N}$ it \mathbb{P} -a.s. holds that*

$$\begin{aligned} H_t^N(i) &= H_0^N(i) + \int_0^t \kappa_H^N \sum_{j \in \mathcal{D}} m(i, j) H_s^N(j) + (\lambda - \kappa_H^N - \alpha^N F_s^N(i)) H_s^N(i) - \frac{\lambda}{K} (H_s^N(i))^2 \\ &\quad - \delta P_s^N(i) H_s^N(i) + \iota_H^N ds + \int_0^t \sqrt{\beta_H^N H_s^N(i)} dW_s^{H,N}(i), \end{aligned} \quad (12)$$

$$\begin{aligned} F_t^N(i) &= F_0^N(i) + \int_0^t \kappa_H^N \sum_{j \in \mathcal{D}} m(i, j) (F_s^N(j) - F_s^N(i)) \frac{H_s^N(j)}{H_s^N(i)} - \alpha^N F_s^N(i) (1 - F_s^N(i)) ds \\ &\quad + \int_0^t \sqrt{\frac{\beta_H^N F_s^N(i) (1 - F_s^N(i))}{H_s^N(i)}} dW_s^{F,N}(i), \end{aligned} \quad (13)$$

$$\begin{aligned} P_t^N(i) &= P_0^N(i) + \int_0^t \kappa_P^N \sum_{j \in \mathcal{D}} m(i, j) P_s^N(j) - (\kappa_P^N + \nu) P_s^N(i) - \gamma (P_s^N(i))^2 + (\eta - \rho F_s^N(i)) P_s^N(i) H_s^N(i) \\ &\quad + \iota_P^N ds + \int_0^t \sqrt{\beta_P^N P_s^N(i)} dW_s^{P,N}(i). \end{aligned} \quad (14)$$

Proof. For all $t \in [0, \infty)$, all $N \in \mathbb{N}$, and all $i \in \mathcal{D}$ we get $\langle W^{H,N}(i) \rangle_t = \langle W^{F,N}(i) \rangle_t = t$ as well as

$$\langle W^{H,N}(i), W^{F,N}(i) \rangle_t = \int_0^t \frac{\sqrt{A_s^N(i) C_s^N(i)} - \sqrt{A_s^N(i) C_s^N(i)}}{H_s^N(i)} ds = 0. \quad (15)$$

Hence, we see that $W^{H,N}(i)$ and $W^{F,N}(i)$, $N \in \mathbb{N}$, $i \in \mathcal{D}$, are independent Brownian motions. Equation (12) follows from Itô's lemma (e.g., Klenke [19]) and rearranging terms. Furthermore, applying Itô's lemma we see

for all $t \in [0, \infty)$, all $i \in \mathcal{D}$, and all $N \in \mathbb{N}$ that \mathbb{P} -a.s. it holds that

$$\begin{aligned}
F_t^N(i) &= F_0^N(i) + \int_0^t \frac{C_s^N(i)}{(H_s^N(i))^2} \left(\kappa_H^N \sum_{j \in \mathcal{D}} m(i, j) (A_s^N(j) - A_s^N(i)) + A_s^N(i) \left(\lambda \left(1 - \frac{H_s^N(i)}{K} \right) - \delta P_s^N(i) - \alpha^N \right) \right. \\
&\quad \left. + \iota_H^N \frac{A_s^N(i)}{H_s^N(i)} \right) ds + \int_0^t \frac{C_s^N(i)}{(H_s^N(i))^2} \sqrt{\beta_H^N A_s^N(i)} dW_s^A(i) \\
&\quad - \int_0^t \frac{A_s^N(i)}{(H_s^N(i))^2} \left(\kappa_H^N \sum_{j \in \mathcal{D}} m(i, j) (C_s^N(j) - C_s^N(i)) + C_s^N(i) \left(\lambda \left(1 - \frac{H_s^N(i)}{K} \right) - \delta P_s^N(i) \right) \right. \\
&\quad \left. + \iota_H^N \frac{C_s^N(i)}{H_s^N(i)} \right) ds - \int_0^t \frac{A_s^N(i)}{(H_s^N(i))^2} \sqrt{\beta_H^N C_s^N(i)} dW_s^C(i) - \int_0^t \frac{C_s^N(i)}{(H_s^N(i))^3} \beta_H^N A_s^N(i) + \frac{A_s^N(i)}{(H_s^N(i))^3} \beta_H^N C_s^N(i) ds \\
&= F_0^N(i) + \int_0^t \frac{\kappa_H^N}{H_s^N(i)} \sum_{j \in \mathcal{D}} m(i, j) ((1 - F_s^N(i)) F_s^N(j) H_s^N(j) - F_s^N(i) (1 - F_s^N(j)) H_s^N(j)) \\
&\quad - \alpha^N F_s^N(i) (1 - F_s^N(i)) ds + \int_0^t \sqrt{\frac{\beta_H^N F_s^N(i) (1 - F_s^N(i))}{H_s^N(i)}} dW_s^{F, N}(i)
\end{aligned} \tag{16}$$

and (13) follows. Finally, we obtain (14) from the definition of $(H^N)_{N \in \mathbb{N}}$ and $(F^N)_{N \in \mathbb{N}}$. \square

Lemma 2.2. Assume the setting of Section 2.1 and assume that for all $N \in \mathbb{N}$ we have $\iota_H^N \geq \frac{1}{2} \beta_H^N$. Furthermore, assume that we have for all $N \in \mathbb{N}$ and all $i \in \mathcal{D}$ that \mathbb{P} -a.s. $H_0^N(i) > 0$. Then we have

$$\mathbb{P} [H_u^N(i) > 0, \text{ for all } u \in [0, \infty), \text{ all } N \in \mathbb{N}, \text{ and all } i \in \mathcal{D}] = 1. \tag{17}$$

Proof. For every $N, M \in \mathbb{N}$ let $\hat{H}^{N, M} : [0, \infty) \times \mathcal{D} \times \Omega \rightarrow [0, \infty)$ be an adapted process with continuous sample paths that for all $t \in [0, \infty)$ and all $i \in \mathcal{D}$ satisfies \mathbb{P} -a.s.

$$\begin{aligned}
\hat{H}_t^{N, M}(i) &= \hat{H}_0^{N, M}(i) + \int_0^t \left[\hat{H}_s^{N, M}(i) \left(\lambda - \alpha^N - \kappa_H^N - \frac{\lambda}{K} \hat{H}_s^{N, M}(i) - \delta M \right) + \iota_H^N \right] ds \\
&\quad + \int_0^t \sqrt{\beta_H^N \hat{H}_s^{N, M}(i)} dW_s^{H, N}(i)
\end{aligned} \tag{18}$$

with $\hat{H}_0^{N, M}(i) = H_0^N(i)$. Due to Feller's boundary classification (e.g., p. 366 in Ethier and Kurtz [7]) with the assumption that for all $N \in \mathbb{N}$ it holds that $\iota_H^N \geq \frac{1}{2} \beta_H^N$ we have for every $N, M \in \mathbb{N}$ and all $i \in \mathcal{D}$ that

$$\mathbb{P} [\hat{H}_t^{N, M}(i) > 0, \text{ for all } t \in [0, \infty)] = 1. \tag{19}$$

For all $N, M \in \mathbb{N}$, all $i \in \mathcal{D}$, and all $t \in [0, \infty)$ consider the event $A_M^N(i) := \left\{ \sup_{s \in [0, t]} P_s^N(i) \leq M \right\}$. We have for all $N, M \in \mathbb{N}$, all $i \in \mathcal{D}$, and all $t \in [0, \infty)$ that

$$\begin{aligned}
A_M^N(i) &\subseteq A_{M+1}^N(i), \\
\mathbb{P} \left[\bigcup_{M \in \mathbb{N}} A_M^N(i) \right] &= \mathbb{P} \left[\sup_{s \in [0, t]} P_s^N(i) < \infty \right] = 1.
\end{aligned} \tag{20}$$

Using a comparison result due to Ikeda and Watanabe (see e.g., Theorem V.43.1 in Rogers and Williams [34]), we get for all $N, M \in \mathbb{N}$, all $i \in \mathcal{D}$, and all $t \in [0, \infty)$ that

$$\mathbb{P} \left[\exists u \in [0, t] : H_u^N(i) < \hat{H}_u^{N, M}(i), \sup_{s \in [0, t]} P_s^N(i) \leq M \right] = 0. \tag{21}$$

Thus, combining (19), (20), and (21) we obtain for all $N \in \mathbb{N}$, all $i \in \mathcal{D}$, and all $t \in [0, \infty)$ that

$$\begin{aligned} 1 &\geq \mathbb{P} [H_u^N(i) > 0, \text{ for all } u \in [0, t]] = 1 - \mathbb{P} [\exists u \in [0, t] : H_u^N(i) = 0] \\ &\geq 1 - \sum_{M \in \mathbb{N}} \mathbb{P} \left[\exists u \in [0, t] : H_u^N(i) = 0, \sup_{s \in [0, t]} P_s^N(i) \leq M \right] \\ &\geq 1 - \sum_{M \in \mathbb{N}} \mathbb{P} \left[\exists u \in [0, t] : H_u^N(i) < \hat{H}_u^{N, M}(i), \sup_{s \in [0, t]} P_s^N(i) \leq M \right] = 1. \end{aligned} \quad (22)$$

This implies for all $N \in \mathbb{N}$, all $i \in \mathcal{D}$, and all $t \in [0, \infty)$ that $\mathbb{P} [H_u^N(i) > 0, \text{ for all } u \in [0, t]] = 1$, which in turn implies (17). This finishes the proof of Lemma 2.2. \square

Lemma 2.3. *Assume the setting of Section 2.1 and assume that for all $N \in \mathbb{N}$ it holds that $\iota_P^N \geq \frac{1}{2}\beta_P^N$. Furthermore, assume that we have for all $N \in \mathbb{N}$ and all $i \in \mathcal{D}$ that \mathbb{P} -a.s. $P_0^N(i) > 0$. Then we have*

$$\mathbb{P} [P_t^N(i) > 0, \text{ for all } t \in [0, \infty), \text{ all } N \in \mathbb{N}, \text{ and all } i \in \mathcal{D}] = 1. \quad (23)$$

Proof. Analogous to the proof of Lemma 2.2. \square

Lemma 2.4. *Assume the setting of Section 2.1. For all $x = (x_i)_{i \in \mathcal{D}} \in E_2$, all $p \in [1, \infty)$, and all sets $\mathcal{D}' \subseteq \mathcal{D}$ it holds that*

$$\sum_{i \in \mathcal{D}'} \sigma_i \left(\sum_{j \in \mathcal{D}} m(i, j) x_j \right)^p \leq \sum_{i \in \mathcal{D}} c \sigma_i x_i^p. \quad (24)$$

Proof. For any $x = (x_i)_{i \in \mathcal{D}} \in E_2$, any $p \in [1, \infty)$, and any set $\mathcal{D}' \subseteq \mathcal{D}$ we obtain from Jensen's inequality and (3) that $\sum_{i \in \mathcal{D}'} \sigma_i \left(\sum_{j \in \mathcal{D}} m(i, j) x_j \right)^p \leq \sum_{i \in \mathcal{D}} \sigma_i \sum_{j \in \mathcal{D}} m(i, j) x_j^p \leq \sum_{i \in \mathcal{D}} c \sigma_i x_i^p$. \square

2.3 Strong convergence of the spatial stochastic Lotka-Volterra processes

In this section we will show the convergence of the time-rescaled Lotka-Volterra processes as given in (12) and (14). In Lemmas 2.5, 2.6, and 2.7 we will provide bounds for the expected value of the sum (over sets of demes) of functionals of the processes weighted by σ . These are then used in Theorem 2.8 to show a result on the behavior of a spatial analogue of a well-known Lyapunov function (e.g., Dobrinevski and Frey [6]). From that the convergence of the processes follows immediately in Theorem 1.2.

Lemma 2.5. *Assume the setting of Section 2.1 and let $p \in \{1\} \cup [2, \infty)$. Then we have*

$$\begin{aligned} \sup_{N \in \mathbb{N}} \sup_{t \in [0, \infty)} \mathbb{E} \left[\left\| (2\eta H_t^N + \delta P_t^N)^p \right\|_\sigma \right] &\leq \sup_{N \in \mathbb{N}} \mathbb{E} \left[\left\| (2\eta H_0^N + \delta P_0^N)^p \right\|_\sigma \right] \\ &+ \left\| \mathbb{1} \right\|_\sigma \left(\frac{\lambda + (1 - \frac{1}{p} + \frac{c}{p})(\bar{\kappa}_H + \bar{\kappa}_P)}{2 \min \left\{ \frac{1}{2\eta} \frac{\lambda}{K}, \frac{1}{4}, \frac{1}{\delta} \gamma \right\}} \right)^p \left(1 + \sqrt{1 + \frac{4 \min \left\{ \frac{1}{2\eta} \frac{\lambda}{K}, \frac{1}{4}, \frac{1}{\delta} \gamma \right\} [2\eta \bar{\iota}_H + \delta \bar{\iota}_P + (p-1)(2\eta \bar{\beta}_H + \frac{1}{2} \delta \bar{\beta}_P)]}{(\lambda + (1 - \frac{1}{p} + \frac{c}{p})(\bar{\kappa}_H + \bar{\kappa}_P))^2}} \right)^p. \end{aligned} \quad (25)$$

Proof. If we assume $\sup_{N \in \mathbb{N}} \mathbb{E} \left[\left\| (H_0^N + P_0^N)^p \right\|_\sigma \right] = \infty$, then the claim trivially holds. For the remainder of the proof assume $\sup_{N \in \mathbb{N}} \mathbb{E} \left[\left\| (H_0^N + P_0^N)^p \right\|_\sigma \right] < \infty$. Define $\mathcal{D}_0 := \emptyset$ and for every $n \in \mathbb{N}$ let $\mathcal{D}_n \subseteq \mathcal{D}$ be a set with $|\mathcal{D}_n| = \min \{n, |\mathcal{D}|\}$ and $\mathcal{D}_n \supseteq \mathcal{D}_{n-1}$. Define real numbers $c_0 := \min \left\{ \frac{1}{2\eta} \frac{\lambda}{K}, \frac{1}{4}, \frac{1}{\delta} \gamma \right\}$, $c_1 := p [2\eta \bar{\iota}_H + \delta \bar{\iota}_P + (p-1)(2\eta \bar{\beta}_H + \frac{1}{2} \delta \bar{\beta}_P)] \in (0, \infty)$, $c_2 := \lambda p + (p-1+c)(\bar{\kappa}_H + \bar{\kappa}_P) \in (0, \infty)$, $c_3 := c_0 p \left(\sum_{k \in \mathcal{D}} \sigma_k \right)^{-\frac{1}{p}} \in (0, \infty)$, and $c_4 := c_1 \left(\sum_{k \in \mathcal{D}} \sigma_k \right)^{\frac{1}{p}} \in (0, \infty)$. For all $N \in \mathbb{N}$, $t \in [0, \infty)$ define $Y_t^N := 2\eta H_t^N + \delta P_t^N$ and for all $N, n \in \mathbb{N}$ and all $t \in [0, \infty)$ let $M_t^{N, n}$ be a real-valued random variable such that \mathbb{P} -a.s. it holds that

$$M_t^{N, n} = \sum_{i \in \mathcal{D}_n} \sigma_i \left(\int_0^t 2\eta p (Y_u^N(i))^{p-1} \sqrt{\beta_H^N H_u^N(i)} dW_u^{H, N}(i) + \int_0^t \delta p (Y_u^N(i))^{p-1} \sqrt{\beta_P^N P_u^N(i)} dW_u^{P, N}(i) \right). \quad (26)$$

Applying Itô's lemma we get for all $N, n \in \mathbb{N}$ and all $t \in [0, \infty)$ that \mathbb{P} -a.s.

$$\begin{aligned}
& \sum_{i \in \mathcal{D}_n} \sigma_i (Y_t^N(i))^p - \sum_{i \in \mathcal{D}_n} \sigma_i (Y_0^N(i))^p \\
&= \sum_{i \in \mathcal{D}_n} \sigma_i \int_0^t 2\eta p (Y_u^N(i))^{p-1} \left(\kappa_H^N \sum_{j \in \mathcal{D}} m(i, j) H_u^N(j) + (\lambda - \kappa_H^N - \alpha^N F_u^N(i)) H_u^N(i) \right. \\
&\quad \left. - \frac{\lambda}{K} (H_u^N(i))^2 - \delta P_u^N(i) H_u^N(i) + \iota_H^N \right) + \delta p (Y_u^N(i))^{p-1} \left(\kappa_P^N \sum_{j \in \mathcal{D}} m(i, j) P_u^N(j) \right. \\
&\quad \left. - (\kappa_P^N + \nu) P_u^N(i) - \gamma (P_u^N(i))^2 + (\eta - \rho F_u^N(i)) P_u^N(i) H_u^N(i) + \iota_P^N \right) \\
&\quad + \frac{1}{2} 4\eta^2 p(p-1) (Y_u^N(i))^{p-2} \beta_H^N H_u^N(i) + \frac{1}{2} \delta^2 p(p-1) (Y_u^N(i))^{p-2} \beta_P^N P_u^N(i) du + M_t^{N,n}.
\end{aligned} \tag{27}$$

Because $1 \geq c_0 4$, $\frac{\lambda}{K} \geq c_0 2\eta$, and $\gamma \geq \delta c_0$ we get for all $N, n \in \mathbb{N}$ and all $t \in [0, \infty)$ that \mathbb{P} -a.s.

$$\begin{aligned}
& \sum_{i \in \mathcal{D}_n} \sigma_i (Y_t^N(i))^p - \sum_{i \in \mathcal{D}_n} \sigma_i (Y_0^N(i))^p \\
&\leq \sum_{i \in \mathcal{D}_n} \sigma_i \int_0^t p (Y_u^N(i))^{p-1} \left(2\eta \bar{\kappa}_H \sum_{j \in \mathcal{D}} m(i, j) H_u^N(j) + 2\eta \lambda H_u^N(i) - c_0 (2\eta H_u^N(i))^2 \right. \\
&\quad \left. - [\eta \delta + c_0 4\eta \delta] P_u^N(i) H_u^N(i) + 2\eta \bar{\iota}_H \right) \\
&\quad + p (Y_u^N(i))^{p-1} \left(\delta \bar{\kappa}_P \sum_{j \in \mathcal{D}} m(i, j) P_u^N(j) + \lambda \delta P_u^N(i) - c_0 (\delta P_u^N(i))^2 + \eta \delta P_u^N(i) H_u^N(i) + \delta \bar{\iota}_P \right) \\
&\quad + p(p-1) (Y_u^N(i))^{p-2} \left((2\eta \bar{\beta}_H + \frac{1}{2} \delta \bar{\beta}_P) 2\eta H_u^N(i) + (\frac{1}{2} \delta \bar{\beta}_P + 2\eta \bar{\beta}_H) \delta P_u^N(i) \right) du + M_t^{N,n} \\
&= \int_0^t \sum_{i \in \mathcal{D}_n} \sigma_i p (Y_u^N(i))^{p-1} 2\eta \bar{\kappa}_H \sum_{j \in \mathcal{D}} m(i, j) H_u^N(j) + \sum_{i \in \mathcal{D}_n} \sigma_i p (Y_u^N(i))^{p-1} \\
&\quad \left(\lambda (Y_u^N(i)) - c_0 (Y_u^N(i))^2 + (2\eta \bar{\iota}_H + \delta \bar{\iota}_P) \right) + \sum_{i \in \mathcal{D}_n} \sigma_i p (Y_u^N(i))^{p-1} \delta \bar{\kappa}_P \sum_{j \in \mathcal{D}} m(i, j) P_u^N(j) \\
&\quad + \sum_{i \in \mathcal{D}_n} \sigma_i p(p-1) (2\eta \bar{\beta}_H + \frac{1}{2} \delta \bar{\beta}_P) (Y_u^N(i))^{p-1} du + M_t^{N,n}.
\end{aligned} \tag{28}$$

Using Young's inequality and Lemma 2.4 we get for all $N, n \in \mathbb{N}$ and all $t \in [0, \infty)$ that \mathbb{P} -a.s.

$$\begin{aligned}
& \sum_{i \in \mathcal{D}_n} \sigma_i (Y_t^N(i))^p - \sum_{i \in \mathcal{D}_n} \sigma_i (Y_0^N(i))^p \\
&\leq \int_0^t \sum_{i \in \mathcal{D}_n} \sigma_i \frac{p-1}{p} p (Y_u^N(i))^p \bar{\kappa}_H + \sum_{i \in \mathcal{D}} \sigma_i \frac{1}{p} p \bar{\kappa}_H c (2\eta H_u^N(i))^p + \sum_{i \in \mathcal{D}_n} \sigma_i \lambda p (Y_u^N(i))^p \\
&\quad + \sum_{i \in \mathcal{D}_n} \sigma_i c_1 (Y_u^N(i))^{p-1} - \sum_{i \in \mathcal{D}_n} \sigma_i c_0 p (Y_u^N(i))^{p+1} + \sum_{i \in \mathcal{D}_n} \sigma_i \frac{p-1}{p} p (Y_u^N(i))^p \bar{\kappa}_P \\
&\quad + \sum_{i \in \mathcal{D}} \sigma_i \frac{1}{p} p \bar{\kappa}_P c (\delta P_u^N(i))^p du + M_t^{N,n} \\
&\leq \int_0^t \sum_{i \in \mathcal{D}} \sigma_i c_2 (Y_u^N(i))^p + \sum_{i \in \mathcal{D}_n} \sigma_i c_1 (Y_u^N(i))^{p-1} - \sum_{i \in \mathcal{D}_n} \sigma_i c_0 p (Y_u^N(i))^{p+1} du + M_t^{N,n}.
\end{aligned} \tag{29}$$

For $N, n, l \in \mathbb{N}$ define $[0, \infty]$ -valued stopping times

$$\tau_l^{N,n} := \inf \left(\left\{ t \in [0, \infty) : \sum_{i \in \mathcal{D}_n} \sigma_i (Y_t^N(i))^p > l \right\} \cup \infty \right). \quad (30)$$

We now get for all $N, n, l \in \mathbb{N}$ and all $t \in [0, \infty)$ that

$$\begin{aligned} & \mathbb{E} \left[\int_0^{t \wedge \tau_l^{N,n}} \sum_{i \in \mathcal{D}_n} \sigma_i \left[\left(2\eta p (Y_u^N(i))^{p-1} \sqrt{\beta_H^N H_u^N(i)} \right)^2 + \left(\delta p (Y_u^N(i))^{p-1} \sqrt{\beta_P^N P_u^N(i)} \right)^2 \right] du \right] \\ &= \mathbb{E} \left[\int_0^{t \wedge \tau_l^{N,n}} \sum_{i \in \mathcal{D}_n} \sigma_i p^2 \left[2\eta \beta_H^N \left((Y_u^N(i))^{p-1} \sqrt{2\eta H_u^N(i)} \right)^2 + \delta \beta_P^N \left((Y_u^N(i))^{p-1} \sqrt{\delta P_u^N(i)} \right)^2 \right] du \right] \quad (31) \\ &\leq (2\eta \beta_H^N + \delta \beta_P^N) \mathbb{E} \left[\int_0^{t \wedge \tau_l^{N,n}} \sum_{i \in \mathcal{D}_n} \sigma_i p^2 \left[\left(2 (Y_u^N(i))^{\frac{2p-1}{2}} \right)^2 \right] du \right] \end{aligned}$$

Using Young's inequality, we obtain for all $N, n, l \in \mathbb{N}$ and all $t \in [0, \infty)$ that

$$\begin{aligned} & \mathbb{E} \left[\int_0^{t \wedge \tau_l^{N,n}} \sum_{i \in \mathcal{D}_n} \sigma_i \left[\left(2\eta p (Y_u^N(i))^{p-1} \sqrt{\beta_H^N H_u^N(i)} \right)^2 + \left(\delta p (Y_u^N(i))^{p-1} \sqrt{\beta_P^N P_u^N(i)} \right)^2 \right] du \right] \\ &\leq (2\eta \beta_H^N + \delta \beta_P^N) \mathbb{E} \left[\int_0^{t \wedge \tau_l^{N,n}} \sum_{i \in \mathcal{D}_n} \sigma_i \left[4p^2 \left(\frac{2p-1}{2p} (Y_u^N(i))^p + \frac{1}{2p} \right)^2 \right] du \right] \\ &\leq (2\eta \beta_H^N + \delta \beta_P^N) \mathbb{E} \left[\int_0^{t \wedge \tau_l^{N,n}} \sum_{i \in \mathcal{D}_n} \frac{\sigma_i^2}{\min_{k \in \mathcal{D}_n} \sigma_k} \left[\left((2p-1) (Y_u^N(i))^p + 1 \right)^2 \right] du \right] \quad (32) \\ &\leq \frac{2\eta \beta_H^N + \delta \beta_P^N}{\min_{k \in \mathcal{D}_n} \sigma_k} \mathbb{E} \left[\int_0^{t \wedge \tau_l^{N,n}} \left[\left(\sum_{i \in \mathcal{D}_n} \sigma_i \left((2p-1) (Y_u^N(i))^p + 1 \right) \right)^2 \right] du \right] \\ &\leq \frac{2\eta \beta_H^N + \delta \beta_P^N}{\min_{k \in \mathcal{D}_n} \sigma_k} \mathbb{E} \left[\int_0^t \left[\left((2p-1) \sum_{i \in \mathcal{D}_n} \sigma_i (Y_{u \wedge \tau_l^{N,n}}^N(i))^p + \|\underline{1}\|_\sigma \right)^2 \right] du \right] \\ &\leq \frac{2\eta \beta_H^N + \delta \beta_P^N}{\min_{k \in \mathcal{D}_n} \sigma_k} t \left[((2p-1)l + \|\underline{1}\|_\sigma)^2 \right] < \infty. \end{aligned}$$

Hence, we get for all $N, n, l \in \mathbb{N}$ and all $t \in [0, \infty)$ that $\mathbb{E} \left[M_{t \wedge \tau_l^{N,n}}^{N,n} \right] = 0$. From this and (29) and using Tonelli's theorem we see for all $N, n, l \in \mathbb{N}$ and all $t \in [0, \infty)$ that

$$\begin{aligned} & \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i (Y_{t \wedge \tau_l^{N,n}}^N(i))^p + \int_0^{t \wedge \tau_l^{N,n}} c_0 p \sum_{i \in \mathcal{D}_n} \sigma_i (Y_u^N(i))^{p+1} du \right] \\ &\leq \mathbb{E} \left[\|(Y_0^N)^p\|_\sigma \right] + \mathbb{E} \left[\int_0^{t \wedge \tau_l^{N,n}} c_2 \|(Y_u^N)^p\|_\sigma + c_1 \|(Y_u^N)^{p-1}\|_\sigma du \right] \quad (33) \\ &\leq \mathbb{E} \left[\|(Y_0^N)^p\|_\sigma \right] + \int_0^t c_2 \mathbb{E} \left[\|(Y_u^N)^p\|_\sigma \right] + c_1 \mathbb{E} \left[\|(Y_u^N)^{p-1}\|_\sigma \right] du. \end{aligned}$$

For every $N, n \in \mathbb{N}$ the map $[0, \infty) \ni t \mapsto \sum_{i \in \mathcal{D}_n} \sigma_i (Y_t^N(i))^p \in \mathbb{R}$ is \mathbb{P} -a.s. continuous which implies for all $N, n \in \mathbb{N}$ and all $t \in [0, \infty)$ that $\mathbb{P} \left[\lim_{l \rightarrow \infty} \tau_l^{N,n} < t \right] = 0$. From Tonelli's theorem and monotone convergence,

then using Fatou's lemma, and finally applying (33) we see for all $N \in \mathbb{N}$ and all $t \in [0, \infty)$ that

$$\begin{aligned}
& \mathbb{E} \left[\sum_{i \in \mathcal{D}} \sigma_i (Y_t^N(i))^p \right] + \int_0^t c_0 p \mathbb{E} \left[\sum_{i \in \mathcal{D}} \sigma_i (Y_u^N(i))^{p+1} \right] du \\
&= \lim_{n \rightarrow \infty} \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i (Y_t^N(i))^p + \int_0^t c_0 p \sum_{i \in \mathcal{D}_n} \sigma_i (Y_u^N(i))^{p+1} du \right] \\
&= \lim_{n \rightarrow \infty} \mathbb{E} \left[\lim_{l \rightarrow \infty} \left(\sum_{i \in \mathcal{D}_n} \sigma_i (Y_{t \wedge \tau_l^{N,n}}^N(i))^p + \int_0^{t \wedge \tau_l^{N,n}} c_0 p \sum_{i \in \mathcal{D}_n} \sigma_i (Y_u^N(i))^{p+1} du \right) \right] \\
&\leq \lim_{n \rightarrow \infty} \liminf_{l \rightarrow \infty} \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i (Y_{t \wedge \tau_l^{N,n}}^N(i))^p + \int_0^{t \wedge \tau_l^{N,n}} c_0 p \sum_{i \in \mathcal{D}_n} \sigma_i (Y_u^N(i))^{p+1} du \right] \\
&\leq \mathbb{E} \left[\left\| (Y_0^N)^p \right\|_\sigma \right] + \int_0^t c_2 \mathbb{E} \left[\left\| (Y_u^N)^p \right\|_\sigma \right] + c_1 \mathbb{E} \left[\left\| (Y_u^N)^{p-1} \right\|_\sigma \right] du.
\end{aligned} \tag{34}$$

This implies using Jensen's inequality for all $N \in \mathbb{N}$ and all $t \in [0, \infty)$ that we get

$$\begin{aligned}
& \mathbb{E} \left[\left\| (Y_t^N)^p \right\|_\sigma \right] - \mathbb{E} \left[\left\| (Y_0^N)^p \right\|_\sigma \right] \\
&\leq \int_0^t c_2 \mathbb{E} \left[\sum_{i \in \mathcal{D}} \sigma_i (Y_u^N(i))^p \right] + c_1 \mathbb{E} \left[\sum_{i \in \mathcal{D}} \sigma_i (Y_u^N(i))^{p-1} \right] - c_0 p \mathbb{E} \left[\sum_{i \in \mathcal{D}} \sigma_i (Y_u^N(i))^{p+1} \right] du \\
&= \int_0^t c_2 \mathbb{E} \left[\sum_{i \in \mathcal{D}} \sigma_i (Y_u^N(i))^p \right] + \frac{\sum_{k \in \mathcal{D}} \sigma_k}{\sum_{l \in \mathcal{D}} \sigma_l} \left(c_1 \mathbb{E} \left[\sum_{i \in \mathcal{D}} \sigma_i (Y_u^N(i))^{p-1} \right] - c_0 p \mathbb{E} \left[\sum_{i \in \mathcal{D}} \sigma_i (Y_u^N(i))^{p+1} \right] \right) du \\
&\leq \int_0^t c_2 \mathbb{E} \left[\sum_{i \in \mathcal{D}} \sigma_i (Y_u^N(i))^p \right] + c_4 \left(\mathbb{E} \left[\sum_{i \in \mathcal{D}} \sigma_i (Y_u^N(i))^p \right] \right)^{\frac{p-1}{p}} - c_3 \left(\mathbb{E} \left[\sum_{i \in \mathcal{D}} \sigma_i (Y_u^N(i))^p \right] \right)^{\frac{p+1}{p}} du \\
&= \int_0^t \left(\mathbb{E} \left[\left\| (Y_u^N)^p \right\|_\sigma \right] \right)^{\frac{p-1}{p}} \left\{ c_4 + c_2 \left(\mathbb{E} \left[\left\| (Y_u^N)^p \right\|_\sigma \right] \right)^{\frac{1}{p}} - c_3 \left(\mathbb{E} \left[\left\| (Y_u^N)^p \right\|_\sigma \right] \right)^{\frac{2}{p}} \right\} du.
\end{aligned} \tag{35}$$

For every $N \in \mathbb{N}$ let $z^N: [0, \infty) \rightarrow \mathbb{R}$ be a process that for all $t \in [0, \infty)$ satisfies

$$z_t^N = z_0^N + \int_0^t (z_s^N)^{\frac{p-1}{p}} \left\{ c_4 + c_2 (z_s^N)^{\frac{1}{p}} - c_3 (z_s^N)^{\frac{2}{p}} \right\} ds \tag{36}$$

with $z_0^N = \mathbb{E} \left[\left\| (Y_0^N)^p \right\|_\sigma \right]$, where uniqueness follows from local Lipschitz continuity. Using classical comparison results from the theory of ODEs, the above computation shows that for all $N \in \mathbb{N}$ and all $t \in [0, \infty)$ we have $\mathbb{E} \left[\left\| (Y_t^N)^p \right\|_\sigma \right] \leq z_t^N$ and for all $N \in \mathbb{N}$ we have $\sup_{t \in [0, \infty)} z_t^N = \max \left\{ \mathbb{E} \left[\left\| (Y_0^N)^p \right\|_\sigma \right], \left(\frac{c_2}{2c_3} + \sqrt{\frac{(c_2)^2}{4c_3^2} + \frac{c_4}{c_3}} \right)^p \right\}$. We thereby conclude that

$$\begin{aligned}
& \sup_{N \in \mathbb{N}} \sup_{t \in [0, \infty)} \mathbb{E} \left[\left\| (2\eta H_t^N + \delta P_t^N)^p \right\|_\sigma \right] \leq \sup_{N \in \mathbb{N}} \sup_{t \in [0, \infty)} z_t^N \leq \sup_{N \in \mathbb{N}} \mathbb{E} \left[\left\| (Y_0^N)^p \right\|_\sigma \right] + \left(\frac{c_2}{2c_3} + \sqrt{\frac{c_2^2}{4c_3^2} + \frac{c_4}{c_3}} \right)^p \\
&= \sup_{N \in \mathbb{N}} \mathbb{E} \left[\left\| (Y_0^N)^p \right\|_\sigma \right] + \frac{c_2}{2c_3} \left(1 + \sqrt{1 + \frac{c_4 c_3}{c_2^2}} \right)^p = \sup_{N \in \mathbb{N}} \mathbb{E} \left[\left\| (2\eta H_0^N + \delta P_0^N)^p \right\|_\sigma \right] \\
&\quad + \left\| \mathbb{1} \right\|_\sigma \left(\frac{\lambda + (1 - \frac{1}{p} + \frac{\varepsilon}{p})(\bar{\kappa}_H + \bar{\kappa}_P)}{2 \min \left\{ \frac{1}{2\eta} \frac{\lambda}{K}, \frac{1}{4}, \frac{1}{\delta} \gamma \right\}} \right)^p \left(1 + \sqrt{1 + \frac{4 \min \left\{ \frac{1}{2\eta} \frac{\lambda}{K}, \frac{1}{4}, \frac{1}{\delta} \gamma \right\} [2\eta \bar{\iota}_H + \delta \bar{\iota}_P + (p-1)(2\eta \bar{\beta}_H + \frac{1}{2} \delta \bar{\beta}_P)]}{(\lambda + (1 - \frac{1}{p} + \frac{\varepsilon}{p})(\bar{\kappa}_H + \bar{\kappa}_P))^2}} \right)^p.
\end{aligned} \tag{37}$$

This finishes the proof of Lemma 2.5. \square

Lemma 2.6. Assume the setting of Section 2.1 and assume $\gamma \geq 2\delta$. Furthermore, assume that for all $N \in \mathbb{N}$ we have $\alpha^N + \kappa_H^N \leq \frac{\lambda}{4}$, $\iota_P^N \leq \frac{\lambda(\nu+\lambda)}{8\delta}$, and $\iota_H^N \geq \frac{4\delta\kappa_P^N}{3(\nu+\lambda)} + \frac{3}{2}\beta_H^N$. Let $\hat{\mathcal{D}} \subseteq \mathcal{D}$ be a set. Then we have

$$\begin{aligned} \sup_{N \in \mathbb{N}} \sup_{t \in [0, \infty)} \mathbb{E} \left[\sum_{i \in \hat{\mathcal{D}}} \sigma_i \left(\frac{2}{\lambda+\nu} \frac{P_t^N(i)}{(H_t^N(i))^2} + \frac{1}{2\delta} \frac{1}{(H_t^N(i))^2} \right) \right] &\leq \sup_{N \in \mathbb{N}} \mathbb{E} \left[\sum_{i \in \hat{\mathcal{D}}} \sigma_i \left(\frac{2}{\lambda+\nu} \frac{P_0^N}{(H_0^N)^2} + \frac{1}{2\delta} \frac{1}{(H_0^N)^2} \right) \right] \\ &+ \frac{4\bar{\kappa}_{PC}}{3\lambda(\lambda+\nu)} \sup_{N \in \mathbb{N}} \sup_{t \in [0, \infty)} \mathbb{E} \left[\sum_{i \in \hat{\mathcal{D}}} \sigma_i (P_t^N(i))^3 \right] + \frac{4}{\lambda(\lambda+\nu)} \left(\frac{\eta^2}{\lambda} + \frac{4\lambda}{K^2} \right) \sup_{N \in \mathbb{N}} \sup_{t \in [0, \infty)} \mathbb{E} \left[\sum_{i \in \hat{\mathcal{D}}} \sigma_i P_t^N(i) \right] + \frac{2}{K^2\delta}. \end{aligned} \quad (38)$$

Proof. If the right-hand side of (38) is infinite, then the claim trivially holds. For the remainder of the proof assume the right-hand side of (38) to be finite. Define $\mathcal{D}_0 := \emptyset$ and for every $n \in \mathbb{N}$ let $\mathcal{D}_n \subseteq \hat{\mathcal{D}}$ be a set with $|\mathcal{D}_n| = \min\{n, |\hat{\mathcal{D}}|\}$ and $\mathcal{D}_n \supseteq \mathcal{D}_{n-1}$. Define $c_1 := \frac{1}{\lambda+\nu}$ and for all $n \in \mathbb{N}$ let

$$c_0^n := \frac{2c_1\bar{\kappa}_{PC}}{3} \sup_{N \in \mathbb{N}} \sup_{t \in [0, \infty)} \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i (P_t^N(i))^3 \right] + 2c_1 \left[\frac{\eta^2}{\lambda} + \frac{4\lambda}{K^2} \right] \sup_{N \in \mathbb{N}} \sup_{t \in [0, \infty)} \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i P_t^N(i) \right] + \frac{\lambda}{K^2\delta}. \quad (39)$$

For $N, n, l \in \mathbb{N}$ define $[0, \infty]$ -valued stopping times

$$\tau_l^{N,n} := \inf \left(\left\{ t \in [0, \infty) : \sum_{i \in \mathcal{D}_n} \sigma_i \left(P_t^N(i) + (H_t^N(i))^{-1} \right) > l \right\} \cup \infty \right). \quad (40)$$

We infer from Lemma 2.2 that for all $N, n \in \mathbb{N}$ the map $[0, \infty) \ni t \mapsto \sum_{i \in \mathcal{D}_n} \sigma_i \left(P_t^N(i) + (H_t^N(i))^{-1} \right) \in \mathbb{R}$ is \mathbb{P} -a.s. continuous. Thereby, we have for all $t \in [0, \infty)$ and all $N, n \in \mathbb{N}$ that

$$\mathbb{P} \left[\lim_{l \rightarrow \infty} \tau_l^{N,n} < t \right] = 0. \quad (41)$$

For all $t \in [0, \infty)$, $N, n, l \in \mathbb{N}$ applying Young's inequality we get

$$\begin{aligned} &\mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \int_0^{t \wedge \tau_l^{N,n}} \left(2c_1 \frac{\sqrt{\beta_P^N P_u^N(i)}}{(H_u^N(i))^2} \right)^2 du \right] \\ &\leq \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \frac{\sigma_i^5}{\min_{k \in \mathcal{D}_n} \{\sigma_k^4\}} t \sup_{u \in [0, t]} 4c_1^2 \bar{\beta}_P \left(\frac{1}{5} \left(P_{u \wedge \tau_l^{N,n}}^N(i) \right)^5 + \frac{4}{5} \left(H_{u \wedge \tau_l^{N,n}}^N(i) \right)^{-5} \right) \right] \\ &\leq \frac{t4c_1^2 \bar{\beta}_P}{\min_{k \in \mathcal{D}_n} \{\sigma_k^4\}} \mathbb{E} \left[\sup_{u \in [0, t]} \left(\sum_{i \in \mathcal{D}_n} \sigma_i \left(P_{u \wedge \tau_l^{N,n}}^N(i) + \left(H_{u \wedge \tau_l^{N,n}}^N(i) \right)^{-1} \right)^5 \right) \right] \leq \frac{t4c_1^2 \bar{\beta}_P}{\min_{k \in \mathcal{D}_n} \{\sigma_k^4\}} t^5 < \infty \end{aligned} \quad (42)$$

and

$$\begin{aligned} &\mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \int_0^{t \wedge \tau_l^{N,n}} \left(\left(4c_1 P_u^N(i) + \frac{1}{\delta} \right) \frac{\sqrt{\beta_H^N H_u^N(i)}}{(H_u^N(i))^3} \right)^2 du \right] \\ &\leq \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \frac{\sigma_i^7}{\min_{k \in \mathcal{D}_n} \{\sigma_k^6\}} t \sup_{u \in [0, t]} \left(4c_1 + \frac{1}{\delta} \right)^2 \bar{\beta}_H \left(\frac{2}{7} \left(P_{u \wedge \tau_l^{N,n}}^N(i) + 1 \right)^7 + \frac{5}{7} \left(H_{u \wedge \tau_l^{N,n}}^N(i) \right)^{-7} \right) \right] \\ &\leq \frac{t(4c_1 + \frac{1}{\delta})^2 \bar{\beta}_H}{\min_{k \in \mathcal{D}_n} \{\sigma_k^6\}} \mathbb{E} \left[\sup_{u \in [0, t]} \left(\sum_{i \in \mathcal{D}_n} \sigma_i \left(P_{u \wedge \tau_l^{N,n}}^N(i) + 1 + \left(H_{u \wedge \tau_l^{N,n}}^N(i) \right)^{-1} \right)^7 \right) \right] \\ &\leq \frac{t(4c_1 + \frac{1}{\delta})^2 \bar{\beta}_H}{\min_{k \in \mathcal{D}_n} \{\sigma_k^6\}} (l + \|\mathbb{1}\|_\sigma)^7 < \infty. \end{aligned} \quad (43)$$

Hence, we obtain for all $t \in [0, \infty)$ and all $N, n, l \in \mathbb{N}$ that

$$\begin{aligned} \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \int_0^{t \wedge \tau_l^{N,n}} 2c_1 \frac{\sqrt{\beta_P^N P_u^N(i)}}{(H_u^N(i))^2} dW_u^{P,N}(i) \right] &= 0, \\ \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \int_0^{t \wedge \tau_l^{N,n}} \left(4c_1 P_u^N(i) + \frac{1}{\delta} \right) \frac{\sqrt{\beta_H^N H_u^N(i)}}{(H_u^N(i))^3} dW_u^{H,N}(i) \right] &= 0. \end{aligned} \quad (44)$$

Define the function $y: \mathbb{N} \times \mathbb{N} \times [0, \infty) \rightarrow [0, \infty]$ by

$$\mathbb{N} \times \mathbb{N} \times [0, \infty) \ni (N, n, t) \mapsto y_t^{N,n} := \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \left(2c_1 \frac{P_u^N(i)}{(H_u^N(i))^2} + \frac{1}{2\delta} \frac{1}{(H_u^N(i))^2} \right) \right]. \quad (45)$$

Recall from the beginning of the proof that we assume for all $N, n \in \mathbb{N}$ that $y_0^{N,n} < \infty$. Now, applying Itô's lemma and using (44), we obtain for all $t \in [0, \infty)$ and all $N, n, l \in \mathbb{N}$ that

$$\begin{aligned} &\mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \left(2c_1 \frac{P_u^N(i)}{(H_u^N(i))^2} + \frac{1}{2\delta} \frac{1}{(H_u^N(i))^2} \right) \right] - y_0^{N,n} \\ &= \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \int_0^{t \wedge \tau_l^{N,n}} 2c_1 \frac{1}{(H_u^N(i))^2} \left(\kappa_P^N \sum_{j \in \mathcal{D}} m(i, j) P_u^N(j) - (\kappa_P^N + \nu) P_u^N(i) - \gamma (P_u^N(i))^2 \right. \right. \\ &\quad \left. \left. + (\eta - \rho F_u^N(i)) P_u^N(i) H_u^N(i) + \iota_P^N \right) - \left(2c_1 \frac{2P_u^N(i)}{(H_u^N(i))^3} + \frac{1}{2\delta} \frac{2}{(H_u^N(i))^3} \right) \left(\kappa_H^N \sum_{j \in \mathcal{D}} m(i, j) H_u^N(j) \right. \right. \\ &\quad \left. \left. + (-\kappa_H^N + \lambda - \alpha^N F_u^N(i)) H_u^N(i) - \frac{\lambda}{K} (H_u^N(i))^2 - \delta P_u^N(i) H_u^N(i) + \iota_H^N \right) \right. \\ &\quad \left. + \frac{1}{2} 2c_1 \frac{6P_u^N(i)}{(H_u^N(i))^4} \beta_H^N H_u^N(i) + \frac{1}{2} \frac{1}{2\delta} \frac{6}{(H_u^N(i))^4} \beta_H^N H_u^N(i) du \right]. \end{aligned} \quad (46)$$

Dropping some negative terms, we now get for all $t \in [0, \infty)$ and all $N, n, l \in \mathbb{N}$ that

$$\begin{aligned} &\mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \left(2c_1 \frac{P_u^N(i)}{(H_u^N(i))^2} + \frac{1}{2\delta} \frac{1}{(H_u^N(i))^2} \right) \right] - y_0^{N,n} \\ &\leq \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \int_0^{t \wedge \tau_l^{N,n}} \frac{2c_1}{(H_u^N(i))^2} \left(\kappa_P^N \sum_{j \in \mathcal{D}} m(i, j) P_u^N(j) - \nu P_u^N(i) - \gamma (P_u^N(i))^2 + \eta P_u^N(i) H_u^N(i) + \iota_P^N \right) \right. \\ &\quad \left. - \left(4c_1 \frac{P_u^N(i)}{(H_u^N(i))^3} + \frac{1}{\delta} \frac{1}{(H_u^N(i))^3} \right) \left((-\kappa_H^N + \lambda - \alpha^N) H_u^N(i) - \frac{\lambda}{K} (H_u^N(i))^2 - \delta P_u^N(i) H_u^N(i) + \iota_H^N \right) \right. \\ &\quad \left. + 6c_1 \frac{P_u^N(i)}{(H_u^N(i))^3} \beta_H^N + \frac{3}{2\delta} \frac{1}{(H_u^N(i))^3} \beta_H^N du \right] \\ &= \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \int_0^{t \wedge \tau_l^{N,n}} 2c_1 \left(\kappa_P^N \frac{1}{(H_u^N(i))^2} \sum_{j \in \mathcal{D}} m(i, j) P_u^N(j) - \nu \frac{P_u^N(i)}{(H_u^N(i))^2} - \gamma \frac{(P_u^N(i))^2}{(H_u^N(i))^2} + \eta \frac{P_u^N(i)}{H_u^N(i)} \right. \right. \\ &\quad \left. \left. + \iota_P^N \frac{1}{(H_u^N(i))^2} - 2(-\kappa_H^N + \lambda - \alpha^N) \frac{P_u^N(i)}{(H_u^N(i))^2} + 2\frac{\lambda}{K} \frac{P_u^N(i)}{H_u^N(i)} + 2\delta \frac{(P_u^N(i))^2}{(H_u^N(i))^2} - 2\iota_H^N \frac{P_u^N(i)}{(H_u^N(i))^3} \right. \right. \\ &\quad \left. \left. + 3 \frac{P_u^N(i)}{(H_u^N(i))^3} \beta_H^N \right) + \frac{\kappa_H^N - \lambda + \alpha^N}{\delta} \frac{1}{(H_u^N(i))^2} + \frac{\lambda}{K\delta} \frac{1}{H_u^N(i)} + \frac{P_u^N(i)}{(H_u^N(i))^2} - \frac{\iota_H^N}{\delta} \frac{1}{(H_u^N(i))^3} + \frac{3\beta_H^N}{2\delta} \frac{1}{(H_u^N(i))^3} du \right]. \end{aligned} \quad (47)$$

Using Young's inequality as well as Lemma 2.4 we get for all $t \in [0, \infty)$ and all $N, n, l \in \mathbb{N}$ that

$$\begin{aligned}
& \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \left(2c_1 \frac{P_{t \wedge \tau_l^{N,n}}^N(i)}{(H_u^N(i))^{N,n}} + \frac{1}{2\delta} \frac{1}{(H_u^N(i))^{N,n}} \right) \right] - y_0^{N,n} \\
& \leq \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \int_0^{t \wedge \tau_l^{N,n}} 2c_1 \left(\frac{2}{3} \kappa_P^N \frac{1}{(H_u^N(i))^3} + \frac{1}{3} \kappa_P^N c (P_u^N(i))^3 - \nu \frac{P_u^N(i)}{(H_u^N(i))^2} - \gamma \frac{(P_u^N(i))^2}{(H_u^N(i))^2} \right. \right. \\
& \quad + \frac{1}{2} \frac{\lambda}{2\eta} \eta \frac{P_u^N(i)}{(H_u^N(i))^2} + \frac{1}{2} \frac{2\eta}{\lambda} \eta P_u^N(i) + \iota_P^N \frac{1}{(H_u^N(i))^2} - 2(-\kappa_H^N + \lambda - \alpha^N) \frac{P_u^N(i)}{(H_u^N(i))^2} + \frac{1}{2} \frac{K}{4} 2 \frac{\lambda}{K} \frac{P_u^N(i)}{(H_u^N(i))^2} \\
& \quad + \frac{1}{2} \frac{4}{K} 2 \frac{\lambda}{K} P_u^N(i) + 2\delta \frac{(P_u^N(i))^2}{(H_u^N(i))^2} - 2\iota_H^N \frac{P_u^N(i)}{(H_u^N(i))^3} + 3 \frac{P_u^N(i)}{(H_u^N(i))^3} \beta_H^N \left. \right) + \frac{\kappa_H^N - \lambda + \alpha^N}{\delta} \frac{1}{(H_u^N(i))^2} \\
& \quad + \frac{1}{2} \frac{K}{2} \frac{\lambda}{K\delta} \frac{1}{(H_u^N(i))^2} + \frac{1}{2} \frac{2}{K} \frac{\lambda}{K\delta} + \frac{P_u^N(i)}{(H_u^N(i))^2} - \frac{\iota_H^N}{\delta} \frac{1}{(H_u^N(i))^3} + \frac{3\beta_H^N}{2\delta} \frac{1}{(H_u^N(i))^3} du \Big] \\
& = \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \int_0^{t \wedge \tau_l^{N,n}} \left[\frac{4c_1}{3} \kappa_P^N - \frac{1}{\delta} \iota_H^N + \frac{3}{2\delta} \beta_H^N \right] \frac{1}{(H_u^N(i))^3} + \frac{2c_1 \kappa_P^N c}{3} (P_u^N(i))^3 + \left[c_1 \left(-2\nu + \frac{\lambda}{2} \right. \right. \right. \\
& \quad + 4(\kappa_H^N - \lambda + \alpha^N) + \frac{\lambda}{2} \Big) + 1 \Big] \frac{P_u^N(i)}{(H_u^N(i))^2} + 2c_1 [-\gamma + 2\delta] \frac{(P_u^N(i))^2}{(H_u^N(i))^2} + 2c_1 \left[\frac{\eta^2}{\lambda} + \frac{4\lambda}{K^2} \right] P_u^N(i) \\
& \quad + \left[2c_1 \iota_P^N + \frac{\kappa_H^N - \lambda + \alpha^N}{\delta} + \frac{\lambda}{4\delta} \right] \frac{1}{(H_u^N(i))^2} + 2c_1 [-2\iota_H^N + 3\beta_H^N] \frac{P_u^N(i)}{(H_u^N(i))^3} + \frac{\lambda}{K^2 \delta} du \Big].
\end{aligned} \tag{48}$$

Recall $\bar{\kappa}_P = \sup_{N \in \mathbb{N}} \kappa_P^N$ and that for all $N \in \mathbb{N}$ we have $\alpha^N + \kappa_H^N \leq \frac{\lambda}{4}$, $\iota_P^N \leq \frac{\lambda(\nu+\lambda)}{8\delta}$, and $\iota_H^N \geq \frac{4\delta\kappa_P^N}{3(\nu+\lambda)} + \frac{3\beta_H^N}{2}$. Furthermore, note that $\frac{\lambda}{2} \leq \frac{1}{2c_1}$. Together with the assumption that $\gamma \geq 2\delta$ we see for all $t \in [0, \infty)$ and all $N, n, l \in \mathbb{N}$ that

$$\begin{aligned}
& \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \left(2c_1 \frac{P_{t \wedge \tau_l^{N,n}}^N(i)}{(H_u^N(i))^{N,n}} + \frac{1}{2\delta} \frac{1}{(H_u^N(i))^{N,n}} \right) \right] - y_0^{N,n} \\
& \leq \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \int_0^{t \wedge \tau_l^{N,n}} \frac{2c_1}{3} \bar{\kappa}_P c (P_u^N(i))^3 - \frac{P_u^N(i)}{(H_u^N(i))^2} + 2c_1 \left[\frac{\eta^2}{\lambda} + \frac{4\lambda}{K^2} \right] P_u^N(i) - \frac{\lambda}{4\delta} \frac{1}{(H_u^N(i))^2} + \frac{\lambda}{K^2 \delta} du \right] \\
& \leq \int_0^t c_0^n du - \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \int_0^{t \wedge \tau_l^{N,n}} \frac{\lambda}{2} \left(2c_1 \frac{P_u^N(i)}{(H_u^N(i))^2} + \frac{1}{2\delta} \frac{1}{(H_u^N(i))^2} \right) du \right].
\end{aligned} \tag{49}$$

Using Tonelli's theorem, Fatou's lemma, and (41) this implies for all $t \in [0, \infty)$ and all $N, n \in \mathbb{N}$ that

$$\begin{aligned}
y_t^{N,n} + \int_0^t \frac{\lambda}{2} y_u^{N,n} du &= y_t^{N,n} + \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \int_0^t \frac{\lambda}{2} \left(2c_1 \frac{P_u^N(i)}{(H_u^N(i))^2} + \frac{1}{2\delta} \frac{1}{(H_u^N(i))^2} \right) du \right] \\
&\leq \liminf_{l \rightarrow \infty} \left(\mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \left(2c_1 \frac{P_{t \wedge \tau_l^{N,n}}^N(i)}{(H_u^N(i))^{N,n}} + \frac{1}{2\delta} \frac{1}{(H_u^N(i))^{N,n}} \right) \right] \right. \\
&\quad \left. + \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \int_0^{t \wedge \tau_l^{N,n}} \frac{\lambda}{2} \left(2c_1 \frac{P_u^N(i)}{(H_u^N(i))^2} + \frac{1}{2\delta} \frac{1}{(H_u^N(i))^2} \right) du \right] \right) \leq y_0^{N,n} + \int_0^t c_0^n du.
\end{aligned} \tag{50}$$

For every $N, n \in \mathbb{N}$ let $z^{N,n}: [0, \infty) \rightarrow \mathbb{R}$ be a process that for all $t \in [0, \infty)$ satisfies $z_t^{N,n} = z_0^{N,n} + \int_0^t (c_0^n - \frac{\lambda}{2} z_s^{N,n}) ds$ with $z_0^{N,n} = y_0^{N,n}$, where uniqueness follows from local Lipschitz continuity. Due to classical comparison results of the theory of ODEs, the above computation yields for all $t \in [0, \infty)$ and all $N, n \in \mathbb{N}$ that

$y_t^{N,n} \leq z_t^{N,n}$ and for all $N, n \in \mathbb{N}$ that $\sup_{t \in [0, \infty)} z_t^{N,n} = \max \left\{ z_0^{N,n}, \frac{2c_0^n}{\lambda} \right\}$. We obtain for all $n \in \mathbb{N}$ that

$$\begin{aligned} \sup_{N \in \mathbb{N}} \sup_{t \in [0, \infty)} y_t^{N,n} &\leq \sup_{N \in \mathbb{N}} \sup_{t \in [0, \infty)} z_t^{N,n} = \max \left\{ \sup_{N \in \mathbb{N}} z_0^{N,n}, \frac{2c_0^n}{\lambda} \right\} \\ &\leq \sup_{N \in \mathbb{N}} \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \left(2c_1 \frac{P_0^N}{(H_0^N)^2} + \frac{1}{2\delta} \frac{1}{(H_0^N)^2} \right) \right] + \frac{2c_0^n}{\lambda}. \end{aligned} \quad (51)$$

Using monotone convergence we thereby conclude

$$\begin{aligned} \sup_{N \in \mathbb{N}} \sup_{t \in [0, \infty)} \mathbb{E} \left[\sum_{i \in \hat{\mathcal{D}}} \sigma_i \left(\frac{2}{\lambda + \nu} \frac{P_t^N(i)}{(H_t^N(i))^2} + \frac{1}{2\delta} \frac{1}{(H_t^N(i))^2} \right) \right] &\leq \lim_{n \rightarrow \infty} \sup_{N \in \mathbb{N}} \sup_{t \in [0, \infty)} y_t^{N,n} \\ &\leq \lim_{n \rightarrow \infty} \left(\sup_{N \in \mathbb{N}} \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \left(2c_1 \frac{P_0^N}{(H_0^N)^2} + \frac{1}{2\delta} \frac{1}{(H_0^N)^2} \right) \right] + \frac{2c_0^n}{\lambda} \right) \\ &\leq \sup_{N \in \mathbb{N}} \mathbb{E} \left[\sum_{i \in \hat{\mathcal{D}}} \sigma_i \left(\frac{2}{\lambda + \nu} \frac{P_0^N}{(H_0^N)^2} + \frac{1}{2\delta} \frac{1}{(H_0^N)^2} \right) \right] + \frac{4\bar{\kappa}_{PC}}{3\lambda(\lambda + \nu)} \sup_{N \in \mathbb{N}} \sup_{t \in [0, \infty)} \mathbb{E} \left[\sum_{i \in \hat{\mathcal{D}}} \sigma_i (P_t^N(i))^3 \right] \\ &\quad + \frac{4}{\lambda(\lambda + \nu)} \left(\frac{\eta^2}{\lambda} + \frac{4\lambda}{K^2} \right) \sup_{N \in \mathbb{N}} \sup_{t \in [0, \infty)} \mathbb{E} \left[\sum_{i \in \hat{\mathcal{D}}} \sigma_i P_t^N(i) \right] + \frac{2}{K^2\delta}, \end{aligned} \quad (52)$$

finishing the proof. \square

Lemma 2.7. Assume the setting of Section 2.1 and assume $\lambda > \nu$ and $\eta - \rho > \frac{\lambda}{K}$. Furthermore, assume that for all $N \in \mathbb{N}$ we have $\iota_H^N \geq \frac{1}{2}\beta_H^N$, $\kappa_P^N + \kappa_H^N + \alpha^N \leq \frac{\lambda - \nu}{2}$, $\iota_P^N \geq \beta_P^N$, and $\iota_H^N \geq \beta_H^N$. Let $\hat{\mathcal{D}} \subseteq \mathcal{D}$ be a set. Then we have

$$\begin{aligned} \sup_{N \in \mathbb{N}} \sup_{t \in [0, \infty)} \mathbb{E} \left[\sum_{i \in \hat{\mathcal{D}}} \sigma_i \left(\frac{(\eta - \rho) - \frac{\lambda}{K}}{2(\bar{\kappa}_P + \nu)} \frac{1}{P_t^N(i)} + \frac{1}{P_t^N(i)H_t^N(i)} \right) \right] &\leq \sup_{N \in \mathbb{N}} \mathbb{E} \left[\sum_{i \in \hat{\mathcal{D}}} \sigma_i \left(\frac{(\eta - \rho) - \frac{\lambda}{K}}{2(\bar{\kappa}_P + \nu)} \frac{1}{P_0^N(i)} + \frac{1}{P_0^N(i)H_0^N(i)} \right) \right] \\ &\quad + \frac{1}{\min\{\bar{\kappa}_P + \nu, \frac{\lambda - \nu}{2}\}} \left(\gamma \frac{(\eta - \rho) - \frac{\lambda}{K}}{2(\bar{\kappa}_P + \nu)} + (\gamma + \delta) \sup_{N \in \mathbb{N}} \sup_{t \in [0, \infty)} \mathbb{E} \left[\sum_{i \in \hat{\mathcal{D}}} \sigma_i \frac{1}{H_t^N(i)} \right] \right). \end{aligned} \quad (53)$$

Proof. If the right-hand side of (53) is infinite, then the claim trivially holds. For the remainder of the proof assume the right-hand side of (53) to be finite. Define $\mathcal{D}_0 := \emptyset$ and for every $n \in \mathbb{N}$ let $\mathcal{D}_n \subseteq \hat{\mathcal{D}}$ be a set with $|\mathcal{D}_n| = \min\{n, |\hat{\mathcal{D}}|\}$ and $\mathcal{D}_n \supseteq \mathcal{D}_{n-1}$. Define $c_0 := \frac{1}{2(\bar{\kappa}_P + \nu)} [(\eta - \rho) - \frac{\lambda}{K}]$ and for every $n \in \mathbb{N}$ let

$$C^n := \gamma c_0 + \left[\gamma + \delta \right] \sup_{N \in \mathbb{N}} \sup_{t \in [0, \infty)} \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \frac{1}{H_t^N(i)} \right]. \quad (54)$$

Note that due to the assumption $\eta - \rho > \frac{\lambda}{K}$ we have $c_0 \in (0, \infty)$. For all $N, n, l \in \mathbb{N}$ define $[0, \infty]$ -valued stopping times

$$\tau_l^{N,n} := \inf \left(\left\{ t \in [0, \infty) : \sum_{i \in \mathcal{D}_n} \sigma_i \left((P_t^N(i))^{-1} + (H_t^N(i))^{-1} \right) > l \right\} \cup \infty \right). \quad (55)$$

We infer from Lemmas 2.2 and 2.3 that for all $N, n \in \mathbb{N}$ the map $[0, \infty) \ni t \mapsto \sum_{i \in \mathcal{D}_n} \sigma_i \left((P_t^N(i))^{-1} + (H_t^N(i))^{-1} \right) \in \mathbb{R}$ is \mathbb{P} -a.s. continuous which implies that we have for all $t \in [0, \infty)$ and all $N, n \in \mathbb{N}$ that

$$\mathbb{P} \left[\lim_{l \rightarrow \infty} \tau_l^{N,n} < t \right] = 0. \quad (56)$$

For all $t \in [0, \infty)$, $N, n, l \in \mathbb{N}$ applying Young's inequality we see that

$$\begin{aligned}
& \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \int_0^{t \wedge \tau_l^{N,n}} \left(\frac{\sqrt{\beta_P^N P_u^N(i)}}{(P_u^N(i))^2} \left(c_0 + \frac{1}{H_u^N(i)} \right) \right)^2 du \right] \\
& \leq \bar{\beta}_P \mathbb{E} \left[t \sup_{u \in [0, t]} \sum_{i \in \mathcal{D}_n} \frac{\sigma_i^5}{\min_{k \in \mathcal{D}_n} \{\sigma_k^4\}} \left(\frac{3}{5} \left(P_{u \wedge \tau_l^{N,n}}^N(i) \right)^{-5} + \frac{2}{5} \left(c_0 + \left(H_{u \wedge \tau_l^{N,n}}^N(i) \right)^{-1} \right)^5 \right) du \right] \\
& \leq \frac{\bar{\beta}_P}{\min_{k \in \mathcal{D}_n} \{\sigma_k^4\}} \mathbb{E} \left[t \sup_{u \in [0, t]} \left(\sum_{i \in \mathcal{D}_n} \sigma_i \left(\left(P_{u \wedge \tau_l^{N,n}}^N(i) \right)^{-1} + \left(H_{u \wedge \tau_l^{N,n}}^N(i) \right)^{-1} + c_0 \right) \right)^5 \right] \leq \frac{\bar{\beta}_P t (l + c_0 \|1\|_\sigma)^5}{\min_{k \in \mathcal{D}_n} \{\sigma_k^4\}} < \infty
\end{aligned} \tag{57}$$

and

$$\begin{aligned}
& \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \int_0^{t \wedge \tau_l^{N,n}} \left(\frac{\sqrt{\beta_H^N H_u^N(i)}}{(H_u^N(i))^2 P_u^N(i)} \right)^2 du \right] \\
& \leq \bar{\beta}_H \mathbb{E} \left[t \sup_{u \in [0, t]} \sum_{i \in \mathcal{D}_n} \frac{\sigma_i^5}{\min_{k \in \mathcal{D}_n} \{\sigma_k^4\}} \left(\frac{3}{5} \left(H_{u \wedge \tau_l^{N,n}}^N(i) \right)^{-5} + \frac{2}{5} \left(P_{u \wedge \tau_l^{N,n}}^N(i) \right)^{-5} \right) du \right] \\
& \leq t \sup_{u \in [0, t]} \frac{\bar{\beta}_H}{\min_{k \in \mathcal{D}_n} \{\sigma_k^4\}} \mathbb{E} \left[\left(\sum_{i \in \mathcal{D}_n} \sigma_i \left(\left(H_{u \wedge \tau_l^{N,n}}^N(i) \right)^{-1} + \left(P_{u \wedge \tau_l^{N,n}}^N(i) \right)^{-1} \right) \right)^5 \right] \leq \frac{t \bar{\beta}_H l^5}{\min_{k \in \mathcal{D}_n} \{\sigma_k^4\}} < \infty.
\end{aligned} \tag{58}$$

Hence, we obtain for all $t \in [0, \infty)$ and all $N, n, l \in \mathbb{N}$ that

$$\begin{aligned}
& \mathbb{E} \left[\int_0^{t \wedge \tau_l^{N,n}} \sum_{i \in \mathcal{D}_n} \sigma_i \sqrt{\beta_P^N P_t^N(i)} \frac{1}{(P_t^N(i))^2} \left(c_0 + \frac{1}{H_t^N(i)} \right) dW_u^{P,N}(i) \right] = 0, \\
& \mathbb{E} \left[\int_0^{t \wedge \tau_l^{N,n}} \sum_{i \in \mathcal{D}_n} \sigma_i \frac{\sqrt{\beta_H^N H_t^N(i)}}{(H_t^N(i))^2 P_t^N(i)} dW_u^{H,N}(i) \right] = 0.
\end{aligned} \tag{59}$$

Define the function $y: \mathbb{N} \times \mathbb{N} \times [0, \infty) \rightarrow [0, \infty]$ by

$$\mathbb{N} \times \mathbb{N} \times [0, \infty) \ni (N, n, t) \mapsto y_t^{N,n} := \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \left(c_0 \frac{1}{P_t^N(i)} + \frac{1}{P_t^N(i) H_t^N(i)} \right) \right]. \tag{60}$$

Recall from the beginning of the proof that we assume for all $N, n \in \mathbb{N}$ that $y_0^{N,n} < \infty$. Applying Itô's lemma and using (59), we get for all $t \in [0, \infty)$ and all $N, n, l \in \mathbb{N}$ that

$$\begin{aligned}
& \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \left(c_0 \frac{1}{P_{t \wedge \tau_l^{N,n}}^N(i)} + \frac{1}{P_{t \wedge \tau_l^{N,n}}^N(i) H_{t \wedge \tau_l^{N,n}}^N(i)} \right) \right] - y_0^{N,n} \\
& = \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \int_0^{t \wedge \tau_l^{N,n}} - \left(c_0 \frac{1}{(P_u^N(i))^2} + \frac{1}{(P_u^N(i))^2 H_u^N(i)} \right) \left(\kappa_P^N \sum_{j \in \mathcal{D}} m(i, j) P_u^N(j) \right. \right. \\
& \quad \left. \left. - (\kappa_P^N + \nu) P_u^N(i) - \gamma (P_u^N(i))^2 + (\eta - \rho F_u^N(i)) P_u^N(i) H_u^N(i) + \iota_P^N \right) + \frac{1}{2} c_0 \frac{2}{(P_u^N(i))^3} \beta_P^N P_u^N(i) \right. \\
& \quad \left. + \frac{1}{2} \frac{2}{(P_u^N(i))^3 H_u^N(i)} \beta_P^N P_u^N(i) - \frac{1}{P_u^N(i) (H_u^N(i))^2} \left(\kappa_H^N \sum_{j \in \mathcal{D}} m(i, j) H_u^N(j) + (-\kappa_H^N + \lambda - \alpha^N F_u^N(i)) H_u^N(i) \right. \right. \\
& \quad \left. \left. - \frac{\lambda}{K} (H_u^N(i))^2 - \delta P_u^N(i) H_u^N(i) + \iota_H^N \right) + \frac{1}{2} \frac{2}{P_u^N(i) (H_u^N(i))^3} \beta_H^N H_u^N(i) du \right].
\end{aligned} \tag{61}$$

Dropping some negative terms, we now get for all $t \in [0, \infty)$ and all $N, n, l \in \mathbb{N}$ that

$$\begin{aligned}
& \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \left(c_0 \frac{1}{P_{t \wedge \tau_l^{N,n}}^N(i)} + \frac{1}{P_{t \wedge \tau_l^{N,n}}^N(i) H_{t \wedge \tau_l^{N,n}}^N(i)} \right) \right] - y_0^{N,n} \\
& \leq \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \int_0^{t \wedge \tau_l^{N,n}} - \left(c_0 \frac{1}{(P_u^N(i))^2} + \frac{1}{(P_u^N(i))^2 H_u^N(i)} \right) \left(-(\kappa_P^N + \nu) P_u^N(i) - \gamma (P_u^N(i))^2 \right. \right. \\
& \quad \left. \left. + (\eta - \rho) P_u^N(i) H_u^N(i) + \iota_P^N \right) + c_0 \beta_P^N \frac{1}{(P_u^N(i))^2} + \beta_P^N \frac{1}{(P_u^N(i))^2 H_u^N(i)} + \beta_H^N \frac{1}{P_u^N(i) (H_u^N(i))^2} \right. \\
& \quad \left. - \frac{1}{P_u^N(i) (H_u^N(i))^2} \left((-\kappa_H^N + \lambda - \alpha^N) H_u^N(i) - \frac{\lambda}{K} (H_u^N(i))^2 - \delta P_u^N(i) H_u^N(i) + \iota_H^N \right) du \right] \\
& = \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \int_0^{t \wedge \tau_l^{N,n}} \left((\kappa_P^N + \nu) c_0 \frac{1}{P_u^N(i)} + \gamma c_0 - (\eta - \rho) c_0 \frac{H_u^N(i)}{P_u^N(i)} - \iota_P^N c_0 \frac{1}{(P_u^N(i))^2} + (\kappa_P^N + \nu) \frac{1}{P_u^N(i) H_u^N(i)} \right. \right. \\
& \quad \left. \left. + \gamma \frac{1}{H_u^N(i)} - (\eta - \rho) \frac{1}{P_u^N(i)} - \iota_P^N \frac{1}{(P_u^N(i))^2 H_u^N(i)} + c_0 \beta_P^N \frac{1}{(P_u^N(i))^2} + \beta_P^N \frac{1}{(P_u^N(i))^2 H_u^N(i)} + \beta_H^N \frac{1}{P_u^N(i) (H_u^N(i))^2} \right. \right. \\
& \quad \left. \left. - (-\kappa_H^N + \lambda - \alpha^N) \frac{1}{P_u^N(i) H_u^N(i)} + \frac{\lambda}{K} \frac{1}{P_u^N(i)} + \delta \frac{1}{H_u^N(i)} - \iota_H^N \frac{1}{P_u^N(i) (H_u^N(i))^2} du \right] \\
& = \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \int_0^{t \wedge \tau_l^{N,n}} \left[(\kappa_P^N + \nu) c_0 - (\eta - \rho) + \frac{\lambda}{K} \right] \frac{1}{P_u^N(i)} + \gamma c_0 - (\eta - \rho) c_0 \frac{H_u^N(i)}{P_u^N(i)} \right. \\
& \quad \left. + [-\iota_P^N c_0 + c_0 \beta_P^N] \frac{1}{(P_u^N(i))^2} + [(\kappa_P^N + \nu) - (-\kappa_H^N + \lambda - \alpha^N)] \frac{1}{P_u^N(i) H_u^N(i)} + [\gamma + \delta] \frac{1}{H_u^N(i)} \right. \\
& \quad \left. + [-\iota_P^N + \beta_P^N] \frac{1}{(P_u^N(i))^2 H_u^N(i)} + [\beta_H^N - \iota_H^N] \frac{1}{P_u^N(i) (H_u^N(i))^2} du \right].
\end{aligned} \tag{62}$$

Recall from Section 2.1 that $\bar{\kappa}_P = \sup_{N \in \mathbb{N}} \kappa_P^N$, and from Assumption 1.1 that $\lambda > \nu$, $\eta - \rho > \frac{\lambda}{K}$ and that for all $N \in \mathbb{N}$ we have $\kappa_P^N + \kappa_H^N + \alpha^N \leq \frac{\lambda - \nu}{2}$, $\iota_P^N \geq \beta_P^N$, and $\iota_H^N \geq \beta_H^N$. Hence, we get for all $t \in [0, \infty)$ and all $N, n, l \in \mathbb{N}$ that

$$\begin{aligned}
& \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \left(c_0 \frac{1}{P_{t \wedge \tau_l^{N,n}}^N(i)} + \frac{1}{P_{t \wedge \tau_l^{N,n}}^N(i) H_{t \wedge \tau_l^{N,n}}^N(i)} \right) \right] - y_0^{N,n} \\
& \leq \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \int_0^{t \wedge \tau_l^{N,n}} - (\bar{\kappa}_P + \nu) c_0 \frac{1}{P_u^N(i)} + \gamma c_0 - \frac{\lambda - \nu}{2} \frac{1}{P_u^N(i) H_u^N(i)} + [\gamma + \delta] \frac{1}{H_u^N(i)} du \right] \\
& \leq \int_0^t C^n du - \mathbb{E} \left[\int_0^{t \wedge \tau_l^{N,n}} \min \{ \bar{\kappa}_P + \nu, \frac{\lambda - \nu}{2} \} \sum_{i \in \mathcal{D}_n} \sigma_i \left(c_0 \frac{1}{P_u^N(i)} + \frac{1}{P_u^N(i) H_u^N(i)} \right) du \right].
\end{aligned} \tag{63}$$

Applying Tonelli's theorem, Fatou's lemma, and (56) we obtain for all $t \in [0, \infty)$ and all $N, n \in \mathbb{N}$ that

$$\begin{aligned}
& y_t^{N,n} + \int_0^t \min \{ \bar{\kappa}_P + \nu, \frac{\lambda - \nu}{2} \} y_u^{N,n} du \\
& = y_t^{N,n} + \mathbb{E} \left[\int_0^t \min \{ \bar{\kappa}_P + \nu, \frac{\lambda - \nu}{2} \} \sum_{i \in \mathcal{D}_n} \sigma_i \left(c_0 \frac{1}{P_u^N(i)} + \frac{1}{P_u^N(i) H_u^N(i)} \right) du \right] \\
& \leq \liminf_{l \rightarrow \infty} \left(\mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \left(c_0 \frac{1}{P_{t \wedge \tau_l^{N,n}}^N(i)} + \frac{1}{P_{t \wedge \tau_l^{N,n}}^N(i) H_{t \wedge \tau_l^{N,n}}^N(i)} \right) \right] \right. \\
& \quad \left. + \mathbb{E} \left[\int_0^{t \wedge \tau_l^{N,n}} \min \{ \bar{\kappa}_P + \nu, \frac{\lambda - \nu}{2} \} \sum_{i \in \mathcal{D}_n} \sigma_i \left(c_0 \frac{1}{P_u^N(i)} + \frac{1}{P_u^N(i) H_u^N(i)} \right) du \right] \right) \leq y_0^{N,n} + \int_0^t C^n du.
\end{aligned} \tag{64}$$

For every $N, n \in \mathbb{N}$, let $z^{N,n}: [0, \infty) \rightarrow \mathbb{R}$ be a process that for all $t \in [0, \infty)$ satisfies $z_t^{N,n} = z_0^{N,n} + \int_0^t (C^n - \min\{\bar{\kappa}_P + \nu, \frac{\lambda - \nu}{2}\} z_s^{N,n}) ds$, with $z_0^{N,n} = y_0^{N,n}$, where uniqueness follows from local Lipschitz continuity. Using classical comparison results from the theory of ODEs, the above computation yields for all $t \in [0, \infty)$ and all $N, n \in \mathbb{N}$ that $y_t^{N,n} \leq z_t^{N,n}$ and for all $N, n \in \mathbb{N}$ that $\sup_{t \in [0, \infty)} z_t^{N,n} = \max\left\{z_0^{N,n}, \frac{C^n}{\min\{\bar{\kappa}_P + \nu, \frac{\lambda - \nu}{2}\}}\right\}$. Hence, we obtain for every $n \in \mathbb{N}$ that

$$\begin{aligned} \sup_{N \in \mathbb{N}} \sup_{t \in [0, \infty)} \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \left(c_0 \frac{1}{P_t^N(i)} + \frac{1}{P_t^N(i) H_t^N(i)} \right) \right] &= \sup_{N \in \mathbb{N}} \sup_{t \in [0, \infty)} y_t^{N,n} \leq \sup_{N \in \mathbb{N}} \sup_{t \in [0, \infty)} z_t^{N,n} \\ &= \max \left\{ \sup_{N \in \mathbb{N}} z_0^{N,n}, \frac{C^n}{\min\{\bar{\kappa}_P + \nu, \frac{\lambda - \nu}{2}\}} \right\} \\ &\leq \sup_{N \in \mathbb{N}} \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \left(c_0 \frac{1}{P_0^N(i)} + \frac{1}{P_0^N(i) H_0^N(i)} \right) \right] + \frac{C^n}{\min\{\bar{\kappa}_P + \nu, \frac{\lambda - \nu}{2}\}}. \end{aligned} \quad (65)$$

Using monotone convergence, we thereby conclude that

$$\begin{aligned} \sup_{N \in \mathbb{N}} \sup_{t \in [0, \infty)} \mathbb{E} \left[\sum_{i \in \mathcal{D}} \sigma_i \left(c_0 \frac{1}{P_t^N(i)} + \frac{1}{P_t^N(i) H_t^N(i)} \right) \right] &= \lim_{n \rightarrow \infty} \sup_{N \in \mathbb{N}} \sup_{t \in [0, \infty)} \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \left(c_0 \frac{1}{P_t^N(i)} + \frac{1}{P_t^N(i) H_t^N(i)} \right) \right] \\ &\leq \lim_{n \rightarrow \infty} \left(\sup_{N \in \mathbb{N}} \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \left(c_0 \frac{1}{P_0^N(i)} + \frac{1}{P_0^N(i) H_0^N(i)} \right) \right] + \frac{C^n}{\min\{\bar{\kappa}_P + \nu, \frac{\lambda - \nu}{2}\}} \right) \\ &= \sup_{N \in \mathbb{N}} \mathbb{E} \left[\sum_{i \in \hat{\mathcal{D}}} \sigma_i \left(c_0 \frac{1}{P_0^N(i)} + \frac{1}{P_0^N(i) H_0^N(i)} \right) \right] + \frac{\gamma c_0 + (\gamma + \delta) \sup_{N \in \mathbb{N}} \sup_{t \in [0, \infty)} \mathbb{E} \left[\sum_{i \in \hat{\mathcal{D}}} \sigma_i \frac{1}{H_t^N(i)} \right]}{\min\{\bar{\kappa}_P + \nu, \frac{\lambda - \nu}{2}\}}, \end{aligned} \quad (66)$$

finishing the proof. \square

Theorem 2.8. *Assume the setting of Section 2.1 and let Assumption 1.1 hold. Then for all $(x, y, z) \in (0, \infty)^2 \times [0, 1]$ it holds that*

$$u(x, y, z) := (\eta - \rho z) \left(x - h_\infty(z) - h_\infty(z) \ln \left(\frac{x}{h_\infty(z)} \right) \right) + \delta \left(y - p_\infty(z) - p_\infty(z) \ln \left(\frac{y}{p_\infty(z)} \right) \right) \geq 0. \quad (67)$$

Furthermore, there exists a constant $c_0 \in (0, \infty)$ such that for every set $\hat{\mathcal{D}} \subseteq \mathcal{D}$, for every $N \in \mathbb{N}$, and every $t \in [0, \infty)$ it holds that

$$\begin{aligned} &\mathbb{E} \left[\sum_{i \in \hat{\mathcal{D}}} \sigma_i u \left(H_t^N(i), P_t^N(i), F_t^N(i) \right) \right] \\ &\quad + \int_0^t (\eta - \rho) \frac{\lambda}{K} \mathbb{E} \left[\sum_{i \in \hat{\mathcal{D}}} \sigma_i \left(H_u^N(i) - h_\infty(F_u^N(i)) \right)^2 \right] + \delta \gamma \mathbb{E} \left[\sum_{i \in \hat{\mathcal{D}}} \sigma_i \left(P_u^N(i) - p_\infty(F_u^N(i)) \right)^2 \right] du \\ &\leq \mathbb{E} \left[\sum_{i \in \hat{\mathcal{D}}} \sigma_i u \left(H_0^N(i), P_0^N(i), F_0^N(i) \right) \right] + t c_0 \max \{ \kappa_H^N, \kappa_P^N, \alpha^N, \iota_H^N, \iota_P^N, \beta_H^N, \beta_P^N \}. \end{aligned} \quad (68)$$

Proof. For the remainder of the proof fix a set $\hat{\mathcal{D}} \subseteq \mathcal{D}$. Define $\mathcal{D}_0 := \emptyset$ and for every $n \in \mathbb{N}$ let $\mathcal{D}_n \subseteq \hat{\mathcal{D}}$ be a set with $|\mathcal{D}_n| = \min\{n, |\hat{\mathcal{D}}|\}$ and $\mathcal{D}_n \supseteq \mathcal{D}_{n-1}$. We will first show that for all $(x, y, z) \in (0, \infty)^2 \times [0, 1]$ it holds that $u(x, y, z) \geq 0$. Define for all $x \in (0, \infty)$ the real-valued function $(0, \infty) \ni y \mapsto f_x(y) := x - y - y \ln \left(\frac{x}{y} \right)$. For all $x \in (0, \infty)$ the function f_x has for all $y \in (0, \infty)$ first and second order derivatives $\frac{df_x}{dy}(y) = \ln(y) - \ln(x)$ and $\frac{d^2 f_x}{dy^2}(y) = \frac{1}{y} > 0$. Thus, for all $x \in (0, \infty)$ the function f_x has its global minimum at x with $f_x(x) = 0$. Consequently, for any $(x, y) \in (0, \infty)^2$ we have $f_x(y) \geq f_x(x) = 0$. This shows that for all

$(x, y, z) \in (0, \infty)^2 \times [0, 1]$ we have that $u(x, y, z) \geq 0$. In order to prove the second part of the claim, we will make use of a Lyapunov function that is defined here analogously to the well-known Lyapunov function in the deterministic setting. Define $D_V := \left(l_\sigma^1 \cap (0, \infty)^{\mathcal{D}}\right) \times \left(l_\sigma^1 \cap (0, \infty)^{\mathcal{D}}\right) \times E_1$. For any subset $\hat{\mathcal{D}}' \subseteq \hat{\mathcal{D}}$ define the function $V_{\hat{\mathcal{D}}'}: D_V \rightarrow [0, \infty]$ for any $(h, p, f) \in D_V$ by

$$V_{\hat{\mathcal{D}}'}((h, p, f)) := \sum_{i \in \hat{\mathcal{D}}'} \sigma_i u(h_i, p_i, f_i). \quad (69)$$

Due to the non-negativity of the mapping u , we obtain for any $\hat{\mathcal{D}}' \subseteq \hat{\mathcal{D}}$ and any $z \in D_V$ that $V_{\hat{\mathcal{D}}'}(z) \in [0, \infty]$ is well-defined. From the fact that for all $x \in (0, \infty)$ we have $-\ln(x) \leq \sqrt{\frac{1}{x}} \leq \frac{1}{2} \left(\frac{1}{x} + 1\right)$ as well as the assumption $\sup_{N \in \mathbb{N}} \mathbb{E} \left[\left\| (H_0^N + P_0^N)^4 + \frac{1}{(H_0^N)^2} + \frac{P_0^N}{(H_0^N)^2} + \frac{1}{P_0^N} + \frac{1}{P_0^N H_0^N} \right\|_\sigma \right] < \infty$ we obtain

$$\sup_{N \in \mathbb{N}} \mathbb{E} [V_{\mathcal{D}}(H_0^N, P_0^N, F_0^N)] < \infty. \quad (70)$$

We now calculate the first and second order partial derivatives that we will need in the application of Itô's lemma below. For all $n \in \mathbb{N}$, $z = (h, p, f) \in D_V$, and $i \in \mathcal{D}_n$ we get $\frac{dV_{\mathcal{D}_n}}{dh_i}(z) = \sigma_i(\eta - \rho f_i) \left(1 - \frac{h_\infty(f_i)}{h_i}\right)$, $\frac{d^2 V_{\mathcal{D}_n}}{dh_i^2}(z) = \sigma_i(\eta - \rho f_i) \frac{h_\infty(f_i)}{h_i^2}$, $\frac{dV_{\mathcal{D}_n}}{dp_i}(z) = \sigma_i \delta \left(1 - \frac{p_\infty(f_i)}{p_i}\right)$, and $\frac{d^2 V_{\mathcal{D}_n}}{dp_i^2}(z) = \sigma_i \delta \frac{p_\infty(f_i)}{p_i^2}$ as well as

$$\begin{aligned} \frac{dV_{\mathcal{D}_n}}{df_i}(z) &= \sigma_i \left[-\rho \left(h_i - h_\infty(f_i) - h_\infty(f_i) \ln \left(\frac{h_i}{h_\infty(f_i)} \right) \right) + (\eta - \rho f_i) \left(-h'_\infty(f_i) - h'_\infty(f_i) \ln \left(\frac{h_i}{h_\infty(f_i)} \right) \right. \right. \\ &\quad \left. \left. - \frac{(h_\infty(f_i))^2}{h_i} \frac{-h_i}{(h_\infty(f_i))^2} h'_\infty(f_i) \right) + \delta \left(-p'_\infty(f_i) - p'_\infty(f_i) \ln \left(\frac{p_i}{p_\infty(f_i)} \right) - \frac{(p_\infty(f_i))^2}{p_i} \frac{-p_i}{(p_\infty(f_i))^2} p'_\infty(f_i) \right) \right] \\ &= \sigma_i \left[-\rho \left(h_i - h_\infty(f_i) - h_\infty(f_i) \ln \left(\frac{h_i}{h_\infty(f_i)} \right) \right) - (\eta - \rho f_i) h'_\infty(f_i) \ln \left(\frac{h_i}{h_\infty(f_i)} \right) - \delta p'_\infty(f_i) \ln \left(\frac{p_i}{p_\infty(f_i)} \right) \right] \end{aligned} \quad (71)$$

and

$$\begin{aligned} \frac{d^2 V_{\mathcal{D}_n}}{df_i^2}(z) &= \sigma_i \left[\rho \left(h'_\infty(f_i) + h'_\infty(f_i) \ln \left(\frac{h_i}{h_\infty(f_i)} \right) + \frac{(h_\infty(f_i))^2}{h_i} (-1) \frac{h_i}{(h_\infty(f_i))^2} h'_\infty(f_i) \right) + \rho h'_\infty(f_i) \ln \left(\frac{h_i}{h_\infty(f_i)} \right) \right. \\ &\quad \left. - (\eta - \rho f_i) \left(h''_\infty(f_i) \ln \left(\frac{h_i}{h_\infty(f_i)} \right) + h'_\infty(f_i) \frac{h_\infty(f_i)}{h_i} \frac{-h_i}{(h_\infty(f_i))^2} h'_\infty(f_i) \right) \right. \\ &\quad \left. - \delta \left(p''_\infty(f_i) \ln \left(\frac{p_i}{p_\infty(f_i)} \right) + p'_\infty(f_i) \frac{p_\infty(f_i)}{p_i} \frac{-p_i}{(p_\infty(f_i))^2} p'_\infty(f_i) \right) \right] \\ &= \sigma_i \left[2\rho h'_\infty(f_i) \ln \left(\frac{h_i}{h_\infty(f_i)} \right) - (\eta - \rho f_i) \left(h''_\infty(f_i) \ln \left(\frac{h_i}{h_\infty(f_i)} \right) - (h'_\infty(f_i))^2 \frac{1}{h_\infty(f_i)} \right) \right. \\ &\quad \left. - \delta \left(p''_\infty(f_i) \ln \left(\frac{p_i}{p_\infty(f_i)} \right) - (p'_\infty(f_i))^2 \frac{1}{p_\infty(f_i)} \right) \right]. \end{aligned} \quad (72)$$

Recall that we have for all $x \in [0, 1]$ that $h_\infty(x) = \frac{1}{b(a-x)}$ and $p_\infty(x) = \frac{\lambda}{\delta} \left(1 - \frac{1}{Kb(a-x)}\right)$ and note that the assumption that $\eta - \rho > \frac{\lambda}{K}$ implies for all $x \in [0, 1]$ that $p_\infty(x) > 0$. Therefore, we get for all $x \in [0, 1]$,

$$\begin{aligned} h'_\infty(x) &= \frac{1}{b(a-x)^2} > 0, \quad h''_\infty(x) = \frac{2}{b(a-x)^3} > 0, \\ p'_\infty(x) &= -\frac{\lambda}{\delta Kb(a-x)^2} < 0, \quad p''_\infty(x) = -\frac{2\lambda}{\delta Kb(a-x)^3} < 0. \end{aligned} \quad (73)$$

So h_∞ , h'_∞ , and h''_∞ are strictly monotonically increasing on $[0, 1]$ while p_∞ , p'_∞ , and p''_∞ are strictly monotonically decreasing on $[0, 1]$. Also we have that $\max_{x \in [0, 1]} \delta p_\infty(x) \leq \lambda$. Observe that for all $x \in (0, \infty)$ we have

$|\ln(x)| \leq \sqrt{x} + \frac{1}{\sqrt{x}}$. Together with Young's inequality as well as Lemmas 2.5, 2.6, and 2.7 we get for all $t \in [0, \infty)$ and all $N, n \in \mathbb{N}$ that

$$\begin{aligned} \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \int_0^t \left(\sqrt{\beta_P^N P_u^N(i)} \delta \left(1 - \frac{p_\infty(F_u^N(i))}{P_u^N(i)} \right) \right)^2 du \right] &\leq \bar{\beta}_P \delta^2 \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \int_0^t P_u^N(i) \left(1 + \frac{(p_\infty(0))^2}{(P_u^N(i))^2} \right) du \right] \\ &\leq \bar{\beta}_P \delta^2 \sup_{u \in [0, t]} \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i t \left(P_u^N(i) + \frac{(p_\infty(0))^2}{P_u^N(i)} \right) \right] < \infty \end{aligned} \quad (74)$$

and

$$\begin{aligned} \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \int_0^t \left(\sqrt{\beta_H^N H_u^N(i)} (\eta - \rho F_u^N(i)) \left(1 - \frac{h_\infty(F_u^N(i))}{H_u^N(i)} \right) \right)^2 du \right] \\ \leq \bar{\beta}_H \eta^2 \sup_{u \in [0, t]} \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i t \left(H_u^N(i) + \frac{(h_\infty(1))^2}{H_u^N(i)} \right) \right] < \infty \end{aligned} \quad (75)$$

and

$$\begin{aligned} \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \int_0^t \left(\sqrt{\frac{\beta_H^N F_u^N(i)(1-F_u^N(i))}{H_u^N(i)}} \left(-\rho \left(H_u^N(i) - h_\infty(F_u^N(i)) - h_\infty(F_u^N(i)) \ln \left(\frac{H_u^N(i)}{h_\infty(F_u^N(i))} \right) \right) \right. \right. \right. \\ \left. \left. \left. - (\eta - \rho F_u^N(i)) h'_\infty(F_u^N(i)) \ln \left(\frac{H_u^N(i)}{h_\infty(F_u^N(i))} \right) - \delta p'_\infty(F_u^N(i)) \ln \left(\frac{P_u^N(i)}{p_\infty(F_u^N(i))} \right) \right) \right)^2 du \right] \\ \leq \bar{\beta}_H \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i \int_0^t \frac{1}{H_u^N(i)} \left(\rho H_u^N(i) + \rho h_\infty(1) + \rho h_\infty(1) \left(\frac{\sqrt{H_u^N(i)}}{\sqrt{h_\infty(0)}} + \frac{\sqrt{h_\infty(1)}}{\sqrt{H_u^N(i)}} \right) \right. \right. \\ \left. \left. + \eta h'_\infty(1) \left(\frac{\sqrt{H_u^N(i)}}{\sqrt{h_\infty(0)}} + \frac{\sqrt{h_\infty(1)}}{\sqrt{H_u^N(i)}} \right) + \delta |p'_\infty(1)| \left(\frac{\sqrt{P_u^N(i)}}{\sqrt{p_\infty(1)}} + \frac{\sqrt{p_\infty(0)}}{\sqrt{P_u^N(i)}} \right) \right)^2 du \right] \\ \leq \bar{\beta}_H \sup_{u \in [0, t]} \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i 2^7 \left(\rho^2 H_u^N(i) + \rho^2 (h_\infty(1))^2 \left(\frac{1}{H_u^N(i)} + \frac{1}{h_\infty(0)} + \frac{h_\infty(1)}{(H_u^N(i))^2} \right) \right. \right. \\ \left. \left. + \eta^2 (h'_\infty(1))^2 \left(\frac{1}{h_\infty(0)} + \frac{h_\infty(1)}{(H_u^N(i))^2} \right) + \delta^2 (p'_\infty(1))^2 \left(\frac{1}{p_\infty(1)} \frac{P_u^N(i)}{H_u^N(i)} + \frac{p_\infty(0)}{P_u^N(i) H_u^N(i)} \right) \right) \right] < \infty. \end{aligned} \quad (76)$$

Hence, we obtain for all $t \in [0, \infty)$ and all $N, n \in \mathbb{N}$ that

$$\begin{aligned} \mathbb{E} \left[\int_0^t \sum_{i \in \mathcal{D}_n} \sigma_i \sqrt{\beta_P^N P_u^N(i)} \delta \left(1 - \frac{p_\infty(F_u^N(i))}{P_u^N(i)} \right) dW_u^{P, N}(i) \right] &= 0, \\ \mathbb{E} \left[\int_0^t \sum_{i \in \mathcal{D}_n} \sigma_i \sqrt{\beta_H^N H_u^N(i)} (\eta - \rho F_u^N(i)) \left(1 - \frac{h_\infty(F_u^N(i))}{H_u^N(i)} \right) dW_u^{H, N}(i) \right] &= 0, \\ \mathbb{E} \left[\int_0^t \sum_{i \in \mathcal{D}_n} \sigma_i \sqrt{\frac{\beta_H^N F_u^N(i)(1-F_u^N(i))}{H_u^N(i)}} \left[-\rho \left(H_u^N(i) - h_\infty(F_u^N(i)) - h_\infty(F_u^N(i)) \ln \left(\frac{H_u^N(i)}{h_\infty(F_u^N(i))} \right) \right) \right. \right. \\ \left. \left. - (\eta - \rho F_u^N(i)) h'_\infty(F_u^N(i)) \ln \left(\frac{H_u^N(i)}{h_\infty(F_u^N(i))} \right) - \delta p'_\infty(F_u^N(i)) \ln \left(\frac{P_u^N(i)}{p_\infty(F_u^N(i))} \right) \right] dW_u^{F, N}(i) \right] &= 0. \end{aligned} \quad (77)$$

For all $t \in [0, \infty)$, all $N \in \mathbb{N}$, and all $i \in \mathcal{D}$ define

$$\begin{aligned}
R_t^N(i) := & \max \left\{ \max \left\{ \eta c, \rho c, \rho, c \eta \frac{h'_\infty(1)}{h_\infty(0)}, \eta \frac{h'_\infty(1)}{h_\infty(0)} \right\} H_t^N(i), \eta h_\infty(1), \eta, \right. \\
& \max \left\{ \frac{\eta}{2} h_\infty(1), \rho (h_\infty(1))^2, \frac{\eta}{2} \frac{(h'_\infty(1))^2}{h_\infty(0)}, \frac{\delta}{2} \frac{(p'_\infty(1))^2}{p_\infty(1)} \right\} \frac{1}{H_t^N(i)}, \delta c P_t^N(i), \delta p_\infty(0), \delta, \frac{\delta}{2} \frac{p_\infty(0)}{P_t^N(i)}, \\
& \max \left\{ \frac{1}{2} \rho^2 (h_\infty(1))^2, \frac{3}{4} \rho^{\frac{4}{3}} (h_\infty(1))^2, \frac{3}{4} \left(\eta h'_\infty(1) \sqrt{h_\infty(1)} \right)^{\frac{4}{3}}, \frac{1}{4}, \frac{\eta}{2} h''_\infty(1) h_\infty(1) \right\} \frac{1}{(H_t^N(i))^2}, \\
& \frac{1}{2} c (H_t^N(i))^2, \frac{1}{4} c (H_t^N(i))^4, \frac{1}{2} \left(\frac{\delta |p'_\infty(1)|}{\sqrt{p_\infty(1)}} \right)^2 \frac{P_t^N(i)}{(H_t^N(i))^2}, \frac{1}{2} \delta^2 |p'_\infty(1)|^2 \frac{p_\infty(0)}{P_t^N(i) H_t^N(i)}, \\
& \left. \delta |p'_\infty(1)| \sqrt{\frac{p_\infty(0)}{P_t^N(i)}}, \rho \frac{h'_\infty(1)}{h_\infty(0)}, \frac{\delta}{2} \frac{|p''_\infty(1)|}{p_\infty(1)} \frac{P_t^N(i)}{H_t^N(i)}, \right\}, \\
b^N := & \max \{ \kappa_H^N, \kappa_P^N, \alpha^N, \iota_H^N, \iota_P^N, \beta_H^N, \beta_P^N \}.
\end{aligned} \tag{78}$$

Note that $\lim_{N \rightarrow \infty} b^N = 0$. Define $c_0 := 32 \sup_{M \in \mathbb{N}} \sup_{u \in [0, \infty)} \mathbb{E} [\|R_u^M\|_\sigma]$. Observe that due to Lemmas 2.5, 2.6, and 2.7

we have $c_0 \in (0, \infty)$. For all $t \in [0, \infty)$, all $N \in \mathbb{N}$, and all $a \in \left\{ \eta, \rho, \eta \frac{h'_\infty(1)}{h_\infty(0)} \right\}$ we have that

$$\sum_{i \in \mathcal{D}} \sigma_i a \sum_{j \in \mathcal{D}} m(i, j) H_t^N(j) \leq \sum_{i \in \mathcal{D}} \sigma_i c a H_t^N(i) \leq \sum_{i \in \mathcal{D}} \sigma_i R_t^N(i). \tag{79}$$

Furthermore, we have for all $t \in [0, \infty)$ and all $N \in \mathbb{N}$ that

$$\sum_{i \in \mathcal{D}} \sigma_i \delta \sum_{j \in \mathcal{D}} m(i, j) P_t^N(j) \leq \sum_{i \in \mathcal{D}} \sigma_i \delta c P_t^N(i) \leq \sum_{i \in \mathcal{D}} \sigma_i R_t^N(i). \tag{80}$$

Using Young's inequality and Lemma 2.4 we get for all $t \in [0, \infty)$ and all $N \in \mathbb{N}$ that

$$\begin{aligned}
\sum_{i \in \mathcal{D}} \sigma_i \rho \frac{h_\infty(F_t^N(i))}{H_t^N(i)} \sum_{j \in \mathcal{D}} m(i, j) H_t^N(j) & \leq \sum_{i \in \mathcal{D}} \sigma_i \left(\frac{1}{2} \left(\rho \frac{h_\infty(F_t^N(i))}{H_t^N(i)} \right)^2 + \frac{1}{2} \left(\sum_{j \in \mathcal{D}} m(i, j) H_t^N(j) \right)^2 \right) \\
& \leq \sum_{i \in \mathcal{D}} \sigma_i \left(R_t^N(i) + \frac{1}{2} c (H_t^N(i))^2 \right) \leq \sum_{i \in \mathcal{D}} \sigma_i 2 R_t^N(i),
\end{aligned} \tag{81}$$

and

$$\begin{aligned}
\sum_{i \in \mathcal{D}} \sigma_i \frac{-\delta p'_\infty(F_t^N(i))}{\sqrt{p_\infty(F_t^N(i))}} \frac{\sqrt{P_t^N(i)}}{H_t^N(i)} \sum_{j \in \mathcal{D}} m(i, j) H_t^N(j) & \leq \sum_{i \in \mathcal{D}} \sigma_i \frac{1}{2} \left(\left(\frac{\delta |p'_\infty(1)|}{\sqrt{p_\infty(1)}} \frac{\sqrt{P_t^N(i)}}{H_t^N(i)} \right)^2 + \left(\sum_{j \in \mathcal{D}} m(i, j) H_t^N(j) \right)^2 \right) \\
& \leq \sum_{i \in \mathcal{D}} \sigma_i \left(R_t^N(i) + \frac{1}{2} c (H_t^N(i))^2 \right) \leq \sum_{i \in \mathcal{D}} \sigma_i 2 R_t^N(i),
\end{aligned} \tag{82}$$

and

$$\begin{aligned}
\sum_{i \in \mathcal{D}} \sigma_i (-1) \delta p'_\infty(F_t^N(i)) \frac{\sqrt{p_\infty(F_t^N(i))}}{\sqrt{P_t^N(i) H_t^N(i)}} \sum_{j \in \mathcal{D}} m(i, j) H_t^N(j) \\
\leq \sum_{i \in \mathcal{D}} \sigma_i \left(\frac{1}{2} \delta^2 |p'_\infty(1)|^2 \frac{p_\infty(0)}{P_t^N(i) H_t^N(i)} + \frac{1}{2} \left(\sum_{j \in \mathcal{D}} m(i, j) H_t^N(j) \right)^2 \frac{1}{H_t^N(i)} \right) \\
\leq \sum_{i \in \mathcal{D}} \sigma_i \left(R_t^N(i) + \frac{1}{4} \left(\sum_{j \in \mathcal{D}} m(i, j) H_t^N(j) \right)^4 + \frac{1}{4} \frac{1}{(H_t^N(i))^2} \right) \leq \sum_{i \in \mathcal{D}} \sigma_i \left(R_t^N(i) + \frac{1}{4} c (H_t^N(i))^4 + R_t^N(i) \right) \\
\leq \sum_{i \in \mathcal{D}} \sigma_i 3 R_t^N(i).
\end{aligned} \tag{83}$$

Again using Young's inequality and Lemma 2.4 we get for all $a \in \left\{ \rho(h_\infty(1))^{\frac{3}{2}}, \eta h'_\infty(1) \sqrt{h_\infty(1)} \right\}$, all $t \in [0, \infty)$, and all $N \in \mathbb{N}$ that

$$\begin{aligned} \sum_{i \in \mathcal{D}} \sigma_i a \left(\frac{1}{H_t^N(i)} \right)^{\frac{3}{2}} \sum_{j \in \mathcal{D}} m(i, j) H_t^N(j) &\leq \sum_{i \in \mathcal{D}} \sigma_i \left(\frac{3}{4} a^{\frac{4}{3}} \left(\frac{1}{H_t^N(i)} \right)^2 + \frac{1}{4} \left(\sum_{j \in \mathcal{D}} m(i, j) H_t^N(j) \right)^4 \right) \\ &\leq \sum_{i \in \mathcal{D}} \sigma_i \left(\mathbb{R}_t^N(i) + \frac{1}{4} c (H_t^N(i))^4 \right) \leq \sum_{i \in \mathcal{D}} \sigma_i 2R_t^N(i). \end{aligned} \quad (84)$$

Due to Lemma 2.1 we have that $W^{H,N}(i)$, $W^{F,N}(i)$, $N \in \mathbb{N}$, $i \in \mathcal{D}$, are independent Brownian motions and due to Lemmas 2.5, 2.6, and 2.7 we have for all $t \in [0, \infty)$ and all $N \in \mathbb{N}$ that \mathbb{P} -a.s. $(H_t^N, P_t^N, F_t^N) \in D_V$. Thus, applying Itô's lemma and using (77) we obtain for all $t \in [0, \infty)$ and all $N, n \in \mathbb{N}$ that

$$\begin{aligned} &\mathbb{E} [V_{\mathcal{D}_n}((H_t^N, P_t^N, F_t^N))] - \mathbb{E} [V_{\mathcal{D}_n}((H_0^N, P_0^N, F_0^N))] \\ &= \mathbb{E} \left[\int_0^t \sum_{i \in \mathcal{D}_n} \sigma_i \left((\eta - \rho F_u^N(i)) \left(1 - \frac{h_\infty(F_u^N(i))}{H_u^N(i)} \right) \left\{ \kappa_H^N \sum_{j \in \mathcal{D}} m(i, j) (H_u^N(j) - H_u^N(i)) \right. \right. \right. \\ &\quad + H_u^N(i) \left[\lambda \left(1 - \frac{H_u^N(i)}{K} \right) - \delta P_u^N(i) - \alpha^N F_u^N(i) \right] + \iota_H^N \Big\} + \frac{(\eta - \rho F_u^N(i)) h_\infty(F_u^N(i))}{2 (H_u^N(i))^2} \beta_H^N H_u^N(i) \\ &\quad + \delta \left(1 - \frac{p_\infty(F_u^N(i))}{P_u^N(i)} \right) \left\{ \kappa_P^N \sum_{j \in \mathcal{D}} m(i, j) (P_u^N(j) - P_u^N(i)) \right. \\ &\quad + P_u^N(i) [-\nu - \gamma P_u^N(i) + (\eta - \rho F_u^N(i)) H_u^N(i)] + \iota_P^N \Big\} + \frac{\delta p_\infty(F_u^N(i))}{2 (P_u^N(i))^2} \beta_P^N P_u^N(i) + \left[-\rho (H_u^N(i) \right. \\ &\quad - h_\infty(F_u^N(i)) - h_\infty(F_u^N(i)) \ln \left(\frac{H_u^N(i)}{h_\infty(F_u^N(i))} \right) - (\eta - \rho F_u^N(i)) h'_\infty(F_u^N(i)) \ln \left(\frac{H_u^N(i)}{h_\infty(F_u^N(i))} \right) \\ &\quad - \delta p'_\infty(F_u^N(i)) \ln \left(\frac{P_u^N(i)}{p_\infty(F_u^N(i))} \right) \Big] \left\{ \kappa_H^N \sum_{j \in \mathcal{D}} m(i, j) (F_u^N(j) - F_u^N(i)) \frac{H_u^N(j)}{H_u^N(i)} - \alpha^N F_u^N(i) (1 - F_u^N(i)) \right\} \\ &\quad + \left\{ \rho h'_\infty(F_u^N(i)) \ln \left(\frac{H_u^N(i)}{h_\infty(F_u^N(i))} \right) - \frac{\eta - \rho F_u^N(i)}{2} \left(h''_\infty(F_u^N(i)) \ln \left(\frac{H_u^N(i)}{h_\infty(F_u^N(i))} \right) - \frac{(h'_\infty(F_u^N(i)))^2}{h_\infty(F_u^N(i))} \right) \right. \\ &\quad \left. \left. - \frac{\delta}{2} (p''_\infty(F_u^N(i)) \ln \left(\frac{P_u^N(i)}{p_\infty(F_u^N(i))} \right) - \frac{(p'_\infty(F_u^N(i)))^2}{p_\infty(F_u^N(i))}) \right\} \frac{\beta_H^N F_u^N(i) (1 - F_u^N(i))}{H_u^N(i)} \right) du \Big]. \end{aligned} \quad (85)$$

Note that for all $x \in [0, 1]$ it holds that $0 < \eta - \rho x \leq \eta$. Together with the fact that for all $x \in (0, \infty)$ we have $\ln(x) \leq \sqrt{x}$, $\ln(x) \leq x$, $|\ln(x)| \leq \sqrt{x} + \sqrt{\frac{1}{x}}$, and $|\ln(x)| \leq x + \sqrt{\frac{1}{x}}$ and dropping negative terms, this implies

for all $t \in [0, \infty)$ and all $N, n \in \mathbb{N}$ that

$$\begin{aligned}
& \mathbb{E} [V_{\mathcal{D}_n} ((H_t^N, P_t^N, F_t^N))] - \mathbb{E} [V_{\mathcal{D}_n} ((H_0^N, P_0^N, F_0^N))] \\
& \leq \mathbb{E} \left[\int_0^t \sum_{i \in \mathcal{D}_n} \sigma_i \left(\eta \kappa_H^N \sum_{j \in \mathcal{D}} m(i, j) H_u^N(j) + \eta h_\infty(F_u^N(i)) \kappa_H^N \right. \right. \\
& \quad + (\eta - \rho F_u^N(i)) (H_u^N(i) - h_\infty(F_u^N(i))) \left[\lambda \left(1 - \frac{H_u^N(i)}{K} \right) - \delta P_u^N(i) \right] + \eta h_\infty(F_u^N(i)) \alpha^N + \eta \iota_H^N \\
& \quad + \frac{\eta}{2} h_\infty(F_u^N(i)) \beta_H^N \frac{1}{H_u^N(i)} + \delta \kappa_P^N \sum_{j \in \mathcal{D}} m(i, j) P_u^N(j) + \delta p_\infty(F_u^N(i)) \kappa_P^N \\
& \quad + \delta (P_u^N(i) - p_\infty(F_u^N(i))) [-\nu - \gamma P_u^N(i) + (\eta - \rho F_u^N(i)) H_u^N(i)] + \delta \iota_P^N + \frac{\delta}{2} \frac{p_\infty(F_u^N(i))}{P_u^N(i)} \beta_P^N \\
& \quad + \rho \kappa_H^N \sum_{j \in \mathcal{D}} m(i, j) H_u^N(j) + \rho H_u^N(i) \alpha^N + \rho \frac{h_\infty(F_u^N(i))}{H_u^N(i)} \left(1 + \frac{H_u^N(i)}{h_\infty(F_u^N(i))} \right. \\
& \quad + \left. \sqrt{\frac{h_\infty(F_u^N(i))}{H_u^N(i)}} \right) \kappa_H^N \sum_{j \in \mathcal{D}} m(i, j) H_u^N(j) + \rho h_\infty(F_u^N(i)) \frac{h_\infty(F_u^N(i))}{H_u^N(i)} \alpha^N + \left[\eta h'_\infty(F_u^N(i)) \left(\sqrt{\frac{h_\infty(F_u^N(i))}{H_u^N(i)}} \right. \right. \\
& \quad + \left. \left. \frac{H_u^N(i)}{h_\infty(F_u^N(i))} \right) - \delta p'_\infty(F_u^N(i)) \left(\sqrt{\frac{p_\infty(F_u^N(i))}{P_u^N(i)}} + \sqrt{\frac{p_\infty(F_u^N(i))}{P_u^N(i)}} \right) \right] \kappa_H^N \sum_{j \in \mathcal{D}} m(i, j) \frac{H_u^N(j)}{H_u^N(i)} \\
& \quad + \left[\eta h'_\infty(F_u^N(i)) \frac{H_u^N(i)}{h_\infty(F_u^N(i))} - \delta p'_\infty(F_u^N(i)) \sqrt{\frac{p_\infty(F_u^N(i))}{P_u^N(i)}} \right] \alpha^N \\
& \quad + \rho h'_\infty(F_u^N(i)) \frac{H_u^N(i)}{h_\infty(F_u^N(i))} \frac{\beta_H^N}{H_u^N(i)} + \frac{\eta}{2} \left(h''_\infty(F_u^N(i)) \frac{h_\infty(F_u^N(i))}{H_u^N(i)} + \frac{(h'_\infty(F_u^N(i)))^2}{h_\infty(F_u^N(i))} \right) \frac{\beta_H^N}{H_u^N(i)} \\
& \quad + \left. \frac{\delta}{2} \left(-p''_\infty(F_u^N(i)) \frac{P_u^N(i)}{p_\infty(F_u^N(i))} + \frac{(p'_\infty(F_u^N(i)))^2}{p_\infty(F_u^N(i))} \right) \frac{\beta_H^N}{H_u^N(i)} \right) du \Big].
\end{aligned} \tag{86}$$

Using (79), (80), (81), (82), (83), and (84) we get for all $t \in [0, \infty)$ and all $N, n \in \mathbb{N}$ that

$$\begin{aligned}
& \mathbb{E} [V_{\mathcal{D}_n} ((H_t^N, P_t^N, F_t^N))] - \mathbb{E} [V_{\mathcal{D}_n} ((H_0^N, P_0^N, F_0^N))] \\
& \leq \mathbb{E} \left[\int_0^t \sum_{i \in \mathcal{D}} \sigma_i \left(b^N 32 R_u^N(i) + (\eta - \rho F_u^N(i)) (H_u^N(i) - h_\infty(F_u^N(i))) \left[\lambda \left(1 - \frac{H_u^N(i)}{K} \right) - \delta P_u^N(i) \right] \right. \right. \\
& \quad \left. \left. + \delta (P_u^N(i) - p_\infty(F_u^N(i))) \left[-\nu - \gamma P_u^N(i) + (\eta - \rho F_u^N(i)) H_u^N(i) \right] \right) \right) du \Big].
\end{aligned} \tag{87}$$

Note that for all $x \in [0, 1]$ we have

$$\begin{aligned}
& \delta p_\infty(x) + \frac{\lambda}{K} h_\infty(x) - \lambda = \frac{\delta \lambda K (\eta - \rho x) - \delta \lambda \nu + \lambda \delta \nu + \lambda^2 \gamma}{\lambda \gamma + \delta K (\eta - \rho x)} - \lambda = \frac{\delta K (\eta - \rho x) + \lambda \gamma}{\lambda \gamma + \delta K (\eta - \rho x)} \lambda - \lambda = 0, \\
& \nu - (\eta - \rho x) h_\infty(x) + \gamma p_\infty(x) = \nu - \frac{(\eta - \rho x) K \delta \nu + (\eta - \rho x) K \gamma \lambda - \gamma \lambda K (\eta - \rho x) + \gamma \lambda \nu}{\lambda \gamma + \delta K (\eta - \rho x)} = \nu - \frac{(\eta - \rho x) K \delta + \gamma \lambda}{\lambda \gamma + \delta K (\eta - \rho x)} \nu = 0.
\end{aligned} \tag{88}$$

From (88) we see that for all $t \in [0, \infty)$ and all $N, n \in \mathbb{N}$ it holds that

$$\begin{aligned}
& \mathbb{E} [V_{\mathcal{D}_n} ((H_t^N, P_t^N, F_t^N))] - \mathbb{E} [V_{\mathcal{D}_n} ((H_0^N, P_0^N, F_0^N))] \\
& \leq \mathbb{E} \left[\int_0^t \sum_{i \in \mathcal{D}} \sigma_i \left(b^N 32 R_u^N(i) + (\eta - \rho F_u^N(i)) (H_u^N(i) - h_\infty(F_u^N(i))) \left[\lambda - \frac{\lambda}{K} (H_u^N(i) - h_\infty(F_u^N(i))) \right] \right. \right. \\
& \quad - \delta (P_u^N(i) - p_\infty(F_u^N(i))) - \lambda \Big] + \delta (P_u^N(i) - p_\infty(F_u^N(i))) \left[-\nu - \gamma (P_u^N(i) - p_\infty(F_u^N(i))) \right. \\
& \quad \left. \left. + (\eta - \rho F_u^N(i)) (H_u^N(i) - h_\infty(F_u^N(i))) + \nu \right] \right) du \Big].
\end{aligned} \tag{89}$$

Hence, we obtain for every $N, n \in \mathbb{N}$ and every $t \in [0, \infty)$ that

$$\begin{aligned} & \mathbb{E} [V_{\mathcal{D}_n} (H_t^N, P_t^N, F_t^N)] + \int_0^t (\eta - \rho) \frac{\lambda}{K} \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i (H_u^N(i) - h_\infty(F_u^N(i)))^2 \right] \\ & + \delta \gamma \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i (P_u^N(i) - p_\infty(F_u^N(i)))^2 \right] du \leq \mathbb{E} [V_{\hat{\mathcal{D}}} (H_0^N, P_0^N, F_0^N)] + tb^N 32 \sup_{M \in \mathbb{N}} \sup_{u \in [0, \infty)} \mathbb{E} [\|R_u^M\|_\sigma]. \end{aligned} \quad (90)$$

Applying monotone convergence we now see that for every $N \in \mathbb{N}$ and every $t \in [0, \infty)$ we have

$$\begin{aligned} & \mathbb{E} [V_{\hat{\mathcal{D}}} (H_t^N, P_t^N, F_t^N)] + \int_0^t (\eta - \rho) \frac{\lambda}{K} \mathbb{E} \left[\sum_{i \in \hat{\mathcal{D}}} \sigma_i (H_u^N(i) - h_\infty(F_u^N(i)))^2 \right] \\ & + \delta \gamma \mathbb{E} \left[\sum_{i \in \hat{\mathcal{D}}} \sigma_i (P_u^N(i) - p_\infty(F_u^N(i)))^2 \right] du \\ & = \lim_{n \rightarrow \infty} \left(\mathbb{E} [V_{\mathcal{D}_n} (H_t^N, P_t^N, F_t^N)] + \int_0^t (\eta - \rho) \frac{\lambda}{K} \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i (H_u^N(i) - h_\infty(F_u^N(i)))^2 \right] \right. \\ & \quad \left. + \delta \gamma \mathbb{E} \left[\sum_{i \in \mathcal{D}_n} \sigma_i (P_u^N(i) - p_\infty(F_u^N(i)))^2 \right] du \right) \leq \mathbb{E} [V_{\hat{\mathcal{D}}} (H_0^N, P_0^N, F_0^N)] + tb^N c_0. \end{aligned} \quad (91)$$

The set $\hat{\mathcal{D}} \subseteq \mathcal{D}$ was arbitrarily chosen and thus, this finishes the proof of Theorem 2.8. \square

2.4 Convergence of relative frequency of altruists

2.4.1 A relative compactness condition

For convenience of the reader, we restate Lemma 3.3 of Klenke and Mytnik [20].

Lemma 2.9. *Let \mathcal{D} be a countable set, let $\sigma \in (0, \infty)^\mathcal{D}$ such that $\sum_{i \in \mathcal{D}} \sigma_i < \infty$, and let $l_\sigma^1 := \{z \in \mathbb{R}^\mathcal{D} : \|z\|_\sigma := \sum_{i \in \mathcal{D}} \sigma_i z_i < \infty\}$. A subset $K \subseteq l_\sigma^1$ is relatively compact if and only if*

$$(i) \sup_{x \in K} \|x\|_\sigma < \infty$$

$$(ii) \text{ for every } \varepsilon \in (0, \infty) \text{ there exists a finite subset } \mathcal{E} \subseteq \mathcal{D} \text{ such that } \sup_{x \in K} \|x \mathbb{1}_{\mathcal{D} \setminus \mathcal{E}}\|_\sigma < \varepsilon.$$

Lemma 2.10. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let \mathcal{D} be a countable set, let $\sigma \in (0, \infty)^\mathcal{D}$ such that $\sum_{i \in \mathcal{D}} \sigma_i < \infty$, let $l_\sigma^1 := \{z \in \mathbb{R}^\mathcal{D} : \|z\|_\sigma := \sum_{i \in \mathcal{D}} \sigma_i z_i < \infty\}$, let $E_2 := l_\sigma^1 \cap [0, \infty)^\mathcal{D}$, let I be a set, and let $Z^i : \Omega \rightarrow E_2$, $i \in I$, be a family of random variables. Assume that $\sup_{i \in I} \mathbb{E}[\|Z^i\|_\sigma] < \infty$ and $\inf_{S \subseteq \mathcal{D}, |S| < \infty} \sup_{i \in I} \sum_{k \in \mathcal{D} \setminus S} \sigma_k \mathbb{E}[Z_k^i] = 0$. Then the family $\{Z^i : i \in I\}$ is relatively compact in E_2 .*

Proof. Fix $\varepsilon \in (0, \infty)$. For each $m \in \mathbb{N}$ by assumption there exists a set $\mathcal{S}_{m, \varepsilon} \subseteq \mathcal{D}$ such that

$$\sup_{i \in I} \sum_{k \in \mathcal{D} \setminus \mathcal{S}_{m, \varepsilon}} \sigma_k \mathbb{E}[Z_k^i] < \frac{\varepsilon}{2m^2(m+1)}. \quad (92)$$

Define the set $K_\varepsilon \subseteq E_2$ by

$$K_\varepsilon := \left\{ x \in E_2 : \|x\|_\sigma \leq \frac{2 \sup_{i \in I} \mathbb{E}[\|Z^i\|_\sigma]}{\varepsilon}, \sup_{m \in \mathbb{N}} \left\{ m \sum_{k \in \mathcal{D} \setminus \mathcal{S}_{m, \varepsilon}} \sigma_k |x_k| \right\} \leq 1 \right\}. \quad (93)$$

Due to the Heine-Borel theorem we can apply Lemma 2.9 to obtain relative compactness of K_ε . By Markov's inequality we get

$$\begin{aligned}
\sup_{i \in I} \mathbb{P} \left[Z^i \notin \overline{K_\varepsilon} \right] &\leq \sup_{i \in I} \mathbb{P} \left[Z^i \notin K_\varepsilon \right] \\
&\leq \sup_{i \in I} \mathbb{P} \left[\|Z^i\|_\sigma > \frac{2 \sup_{j \in I} \mathbb{E}[\|Z^j\|_\sigma]}{\varepsilon} \right] + \sup_{i \in I} \sum_{m=1}^{\infty} \mathbb{P} \left[\sum_{k \in \mathcal{D} \setminus \mathcal{S}_{m,\varepsilon}} \sigma_k Z_k^i > \frac{1}{m} \right] \\
&\leq \frac{\varepsilon}{2 \sup_{j \in I} \mathbb{E}[\|Z^j\|_\sigma]} \sup_{i \in I} \mathbb{E}[\|Z^i\|_\sigma] + \sum_{m=1}^{\infty} m \sup_{i \in I} \sum_{k \in \mathcal{D} \setminus \mathcal{S}_{m,\varepsilon}} \sigma_k \mathbb{E}[Z_k^i] \leq \frac{\varepsilon}{2} + \sum_{m=1}^{\infty} m \frac{\varepsilon}{2m^2(m+1)} = \varepsilon.
\end{aligned} \tag{94}$$

Since ε was arbitrarily chosen it follows that $\{Z^i : i \in I\}$ is tight in E_2 . Due to Prohorov's theorem (e.g., Theorem 3.2.2 in Ethier and Kurtz [7]) the claim follows. \square

2.4.2 Proof of Theorem 1.3

Lemma 2.11. *Assume the setting of Section 2.1 and assume that for all $N \in \mathbb{N}$ we have $\sum_{i \in \mathcal{D}} \sigma_i \mathbb{E}[H_0^N(i)] < \infty$. For all $n \in \mathbb{N}$ denote by m^n the n -fold matrix product of m . Then we get for all $t \in [0, \infty)$, all $i \in \mathcal{D}$, and all $N \in \mathbb{N}$ that*

$$\mathbb{E}[H_t^N(i)] \leq \mathbb{E} \left[\sum_{j \in \mathcal{D}} \sum_{n=0}^{\infty} e^{-t\kappa_H^N} \frac{(t\kappa_H^N)^n}{n!} m^n(i, j) H_0^N(j) \right] + \frac{K}{2} \left(1 + \sqrt{1 + \frac{4L_H}{K\lambda}} \right). \tag{95}$$

Proof. We have for every $n \in \mathbb{N}$ and every $i, j \in \mathcal{D}$ that $m^n(i, j) \in [0, 1]$. Hence, we get for all $T \in [0, \infty)$ and all $i, j \in \mathcal{D}$ that

$$\sum_{n=0}^{\infty} \sup_{t \in [0, T]} e^{-t\kappa_H^N} \frac{t^n}{n!} m^n(i, j) < \infty. \tag{96}$$

Thereby, for all $t \in [0, \infty)$ and all $i, j \in \mathcal{D}$ we can define

$$m_t(i, j) := \sum_{n=0}^{\infty} e^{-t\kappa_H^N} \frac{t^n}{n!} m^n(i, j). \tag{97}$$

By (96) and using dominated convergence, we can compute for all $t \in [0, \infty)$ and all $i, j \in \mathcal{D}$ that

$$\begin{aligned}
\frac{d}{dt} m_t(i, j) &= -m_t(i, j) + \sum_{n=1}^{\infty} e^{-t\kappa_H^N} \frac{t^{n-1}}{(n-1)!} m^n(i, j) = -m_t(i, j) + \sum_{n=0}^{\infty} e^{-t\kappa_H^N} \frac{t^n}{n!} m^{n+1}(i, j) \\
&= -m_t(i, j) + \sum_{n=0}^{\infty} e^{-t\kappa_H^N} \frac{t^n}{n!} \sum_{k \in \mathcal{D}} m^n(i, k) m(k, j) = \sum_{k \in \mathcal{D}} m_t(i, k) (m(k, j) - \mathbb{1}_{j=k}).
\end{aligned} \tag{98}$$

Furthermore, note that for all $t \in [0, \infty)$ and all $i \in \mathcal{D}$ we have

$$\sum_{j \in \mathcal{D}} m_t(i, j) = \sum_{j \in \mathcal{D}} \sum_{n=0}^{\infty} e^{-t\kappa_H^N} \frac{t^n}{n!} m^n(i, j) = \sum_{n=0}^{\infty} e^{-t\kappa_H^N} \frac{t^n}{n!} = 1. \tag{99}$$

For all $t \in [0, \infty)$, $s \in [0, t]$, $i \in \mathcal{D}$, $N \in \mathbb{N}$ define

$$Y_s^{N,t}(i) := \sum_{j \in \mathcal{D}} m_{(t-s)\kappa_H^N}(i, j) H_s^N(j). \tag{100}$$

Observe that since for all $i, j \in \mathcal{D}$ it holds that $m_0(i, j) = \mathbb{1}_{i=j}$ we have for all $t \in [0, \infty)$, all $i \in \mathcal{D}$, and all $N \in \mathbb{N}$ that

$$Y_t^{N,t}(i) = H_t^N(i). \tag{101}$$

Furthermore, using (3) we have for all $t \in [0, \infty)$ and all $N \in \mathbb{N}$ that

$$\begin{aligned}
\sum_{i \in \mathcal{D}} \sigma_i \mathbb{E} [Y_0^{N,t}(i)] &= \sum_{i \in \mathcal{D}} \sigma_i \mathbb{E} \left[\sum_{j \in \mathcal{D}} m_{t\kappa_H^N}(i, j) H_0^N(j) \right] = \sum_{i \in \mathcal{D}} \sigma_i \sum_{j \in \mathcal{D}} \sum_{n=0}^{\infty} e^{-t\kappa_H^N \frac{(t\kappa_H^N)^n}{n!}} m^n(i, j) \mathbb{E} [H_0^N(j)] \\
&= \sum_{j \in \mathcal{D}} \sum_{n=0}^{\infty} e^{-t\kappa_H^N \frac{(t\kappa_H^N)^n}{n!}} \mathbb{E} [H_0^N(j)] \sum_{i \in \mathcal{D}} \sigma_i m^n(i, j) \leq \sum_{j \in \mathcal{D}} \sum_{n=0}^{\infty} e^{-t\kappa_H^N \frac{(t\kappa_H^N)^n}{n!}} \mathbb{E} [H_0^N(j)] c^n \sigma_j \\
&= \sum_{j \in \mathcal{D}} \sum_{n=0}^{\infty} e^{-t\kappa_H^N \frac{(t\kappa_H^N c)^n}{n!}} \mathbb{E} [H_0^N(j)] \sigma_j = e^{t\kappa_H^N(c-1)} \mathbb{E} [\|H_0^N\|_{\sigma}].
\end{aligned} \tag{102}$$

For all $t \in [0, \infty)$, $s \in [0, t]$, $i, j \in \mathcal{D}$, $N \in \mathbb{N}$ we see from (98) that we have

$$\frac{d}{ds} m_{(t-s)\kappa_H^N}(i, j) = -\kappa_H^N \sum_{k \in \mathcal{D}} m_{(t-s)\kappa_H^N}(i, k) (m(k, j) - \mathbb{1}_{j=k}). \tag{103}$$

For $t \in [0, \infty)$, $N, l \in \mathbb{N}$, $i \in \mathcal{D}$ define

$$\tau_l^{N,t}(i) := \inf \left(\{u \in [0, t] : Y_u^{N,t}(i) > l\} \cup \infty \right). \tag{104}$$

Using the fact that for all $t \in [0, \infty)$, all $u \in [0, t]$, all $N \in \mathbb{N}$, and all $i, j \in \mathcal{D}$ we have $m_{(t-u)\kappa_H^N}(i, j) \in [0, 1]$ we get for all $t \in [0, \infty)$, all $s \in [0, t]$, all $N, l \in \mathbb{N}$, and all $i \in \mathcal{D}$ that

$$\begin{aligned}
\int_0^{s \wedge \tau_l^{N,t}(i)} \sum_{j \in \mathcal{D}} \left(m_{(t-u)\kappa_H^N}(i, j) \sqrt{\beta_H^N H_u^N(j)} \right)^2 du &\leq \int_0^{s \wedge \tau_l^{N,t}(i)} \sum_{j \in \mathcal{D}} m_{(t-u)\kappa_H^N}(i, j) \beta_H^N H_u^N(j) du \\
&= \int_0^{s \wedge \tau_l^{N,t}(i)} \beta_H^N Y_u^{N,t}(i) du \leq \int_0^s \beta_H^N Y_{u \wedge \tau_l^{N,t}(i)}^{N,t}(i) du \leq t \beta_H^N l.
\end{aligned} \tag{105}$$

For all $t \in [0, \infty)$, $s \in [0, t]$, $i \in \mathcal{D}$, $N \in \mathbb{N}$ using Itô's lemma with (99) and (103) we get \mathbb{P} -a.s.

$$\begin{aligned}
Y_s^{N,t}(i) - Y_0^{N,t}(i) &= \int_0^s \sum_{j \in \mathcal{D}} m_{(t-u)\kappa_H^N}(i, j) \left(\kappa_H^N \sum_{k \in \mathcal{D}} m(j, k) H_u^N(k) + (\lambda - \kappa_H^N - \alpha^N F_t^N(j)) H_u^N(j) \right. \\
&\quad \left. - \frac{\lambda}{K} (H_u^N(j))^2 - \delta H_u^N(j) P_u^N(j) + \iota_H^N \right) - \sum_{j \in \mathcal{D}} \kappa_H^N \sum_{k \in \mathcal{D}} m_{(t-u)\kappa_H^N}(i, k) (m(k, j) - \mathbb{1}_{j=k}) H_u^N(j) du \\
&\quad + \sum_{j \in \mathcal{D}} \int_0^s m_{(t-u)\kappa_H^N}(i, j) \sqrt{\beta_H^N H_u^N(j)} dW_u^{N,H}(j) \\
&= \int_0^s \sum_{j \in \mathcal{D}} m_{(t-u)\kappa_H^N}(i, j) \left((\lambda - \alpha^N F_t^N(j)) H_u^N(j) - \frac{\lambda}{K} (H_u^N(j))^2 - \delta H_u^N(j) P_u^N(j) \right) + \iota_H^N du \\
&\quad + \sum_{j \in \mathcal{D}} \int_0^s m_{(t-u)\kappa_H^N}(i, j) \sqrt{\beta_H^N H_u^N(j)} dW_u^{N,H}(j) \\
&\leq \int_0^s \sum_{j \in \mathcal{D}} m_{(t-u)\kappa_H^N}(i, j) \left(\lambda H_u^N(j) - \frac{\lambda}{K} (H_u^N(j))^2 \right) + \iota_H^N du \\
&\quad + \sum_{j \in \mathcal{D}} \int_0^s m_{(t-u)\kappa_H^N}(i, j) \sqrt{\beta_H^N H_u^N(j)} dW_u^{N,H}(j).
\end{aligned} \tag{106}$$

Thus, using (105) and (106) we get for all $t \in [0, \infty)$, all $s \in [0, t]$, all $i \in \mathcal{D}$, and all $N, l \in \mathbb{N}$ that

$$\begin{aligned} \mathbb{E} \left[Y_{s \wedge \tau_l^{N,t}(i)}^{N,t}(i) \right] - \mathbb{E} \left[Y_0^{N,t}(i) \right] &\leq \mathbb{E} \left[\int_0^{s \wedge \tau_l^{N,t}(i)} \sum_{j \in \mathcal{D}} m_{(t-u)\kappa_H^N}(i, j) \lambda H_u^N(j) + \iota_H^N du \right] \\ &\leq \mathbb{E} \left[\int_0^s \sum_{j \in \mathcal{D}} m_{(t-u \wedge \tau_l^{N,t}(i))\kappa_H^N}(i, j) \lambda H_{u \wedge \tau_l^{N,t}(i)}^N(j) + \iota_H^N du \right] \\ &= \int_0^s \lambda \mathbb{E} \left[Y_{u \wedge \tau_l^{N,t}(i)}^{N,t}(i) \right] + \iota_H^N du \leq t \iota_H^N + \lambda \int_0^s \mathbb{E} \left[Y_{u \wedge \tau_l^{N,t}(i)}^{N,t}(i) \right] du. \end{aligned} \quad (107)$$

Now, using Gronwall's lemma (e.g., Klenke [19]), we get for all $t \in [0, \infty)$, all $s \in [0, t]$, all $i \in \mathcal{D}$, and all $N, l \in \mathbb{N}$ that

$$\mathbb{E} \left[Y_{s \wedge \tau_l^{N,t}(i)}^{N,t}(i) \right] \leq \left(\mathbb{E} \left[Y_0^{N,t}(i) \right] + t \iota_H^N \right) e^{\lambda s} \leq \left(\mathbb{E} \left[Y_0^{N,t}(i) \right] + t \iota_H^N \right) e^{\lambda t}. \quad (108)$$

For all $t \in [0, \infty)$, $N \in \mathbb{N}$, $i \in \mathcal{D}$ the \mathbb{P} -a.s. continuous paths of $(Y_u^{N,t}(i))_{u \in [0, t]}$ imply $\mathbb{P} \left[\sup_{u \in [0, t]} Y_u^{N,t}(i) < \infty \right] = 1$. Hence, we get for all $t \in [0, \infty)$, all $N \in \mathbb{N}$, and all $i \in \mathcal{D}$ that

$$\mathbb{P} \left[\lim_{l \rightarrow \infty} \tau_l^{N,t}(i) = \infty \right] = 1. \quad (109)$$

Using the assumption that for all $N \in \mathbb{N}$ we have $\sum_{i \in \mathcal{D}} \sigma_i \mathbb{E} \left[H_0^N(i) \right] < \infty$ together with (101), (102), (108), and (109) with Fatou's lemma we obtain for all $t \in [0, \infty)$ and all $N \in \mathbb{N}$ that

$$\begin{aligned} \sum_{i \in \mathcal{D}} \sigma_i \mathbb{E} \left[H_t^N(i) \right] &= \sum_{i \in \mathcal{D}} \sigma_i \mathbb{E} \left[Y_t^{N,t}(i) \right] = \sum_{i \in \mathcal{D}} \sigma_i \mathbb{E} \left[\lim_{l \rightarrow \infty} Y_{t \wedge \tau_l^{N,t}(i)}^{N,t}(i) \right] \leq \sum_{i \in \mathcal{D}} \sigma_i \liminf_{l \rightarrow \infty} \mathbb{E} \left[Y_{t \wedge \tau_l^{N,t}(i)}^{N,t}(i) \right] \\ &\leq \sum_{i \in \mathcal{D}} \sigma_i \liminf_{l \rightarrow \infty} \left(\mathbb{E} \left[Y_0^{N,t}(i) \right] + t \iota_H^N \right) e^{\lambda t} = \sum_{i \in \mathcal{D}} \sigma_i \left(\mathbb{E} \left[Y_0^{N,t}(i) \right] + t \iota_H^N \right) e^{\lambda t} \\ &\leq \left(e^{t \kappa_H^N(c-1)} \mathbb{E} \left[\|H_0^N\|_\sigma \right] + \sum_{i \in \mathcal{D}} \sigma_i t \iota_H^N \right) e^{\lambda t} < \infty. \end{aligned} \quad (110)$$

Using the fact that for all $t \in [0, \infty)$, all $u \in [0, t]$, all $N \in \mathbb{N}$, and all $i, j \in \mathcal{D}$ we have $m_{(t-u)\kappa_H^N}(i, j) \in [0, 1]$ this implies for all $t \in [0, \infty)$, all $s \in [0, t]$, all $N \in \mathbb{N}$, and all $i \in \mathcal{D}$, that

$$\begin{aligned} \mathbb{E} \left[\sum_{j \in \mathcal{D}} \int_0^s \left(m_{(t-u)\kappa_H^N}(i, j) \sqrt{\beta_H^N H_u^N(j)} \right)^2 du \right] &= \int_0^s \mathbb{E} \left[\sum_{j \in \mathcal{D}} \left(m_{(t-u)\kappa_H^N}(i, j) \sqrt{\beta_H^N H_u^N(j)} \right)^2 \right] du \\ &\leq \beta_H^N \int_0^s \mathbb{E} \left[\sum_{j \in \mathcal{D}} m_{(t-u)\kappa_H^N}(i, j) H_u^N(j) \right] du = \beta_H^N \int_0^s \mathbb{E} \left[Y_u^{N,t}(i) \right] du < \infty. \end{aligned} \quad (111)$$

Thus, taking expectations in (106) gives for all $t \in [0, \infty)$, all $s \in [0, t]$, all $i \in \mathcal{D}$, and all $N \in \mathbb{N}$ using Jensen's inequality

$$\begin{aligned} \mathbb{E} \left[Y_s^{N,t}(i) \right] - \mathbb{E} \left[Y_0^{N,t}(i) \right] &\leq \int_0^s \left(\lambda \mathbb{E} \left[Y_u^{N,t}(i) \right] - \frac{\lambda}{K} \mathbb{E} \left[\sum_{j \in \mathcal{D}} m_{(t-u)\kappa_H^N}(i, j) (H_u^N(j))^2 \right] \right) + \iota_H^N du \\ &\leq \int_0^s \left(\lambda \mathbb{E} \left[Y_u^{N,t}(i) \right] - \frac{\lambda}{K} \mathbb{E} \left[(Y_u^{N,t}(i))^2 \right] \right) + \iota_H^N du \\ &\leq \int_0^s \left(\lambda \mathbb{E} \left[Y_u^{N,t}(i) \right] - \frac{\lambda}{K} (\mathbb{E} \left[Y_u^{N,t}(i) \right])^2 \right) + \iota_H^N du. \end{aligned} \quad (112)$$

For $t \in [0, \infty)$, $i \in \mathcal{D}$, $N \in \mathbb{N}$ let $z^{N,t}(i) : [0, \infty) \rightarrow \mathbb{R}$ be a process that for all $s \in [0, \infty)$ satisfies

$$z_s^{N,t}(i) = z_0^{N,t}(i) + \int_0^s \left(\lambda z_u^{N,t}(i) - \frac{\lambda}{K} (z_u^{N,t}(i))^2 + \bar{\iota}_H \right) du \quad (113)$$

with $z_0^{N,t}(i) = \mathbb{E} [Y_0^{N,t}(i)]$ where uniqueness follows from local Lipschitz continuity. Define $c_1 := \frac{K}{2} + \sqrt{\frac{K^2}{4} + \frac{K\bar{\iota}_H}{\lambda}} \in (0, \infty)$. Using classical comparison results from the theory of ODEs, the above computation shows that for all $N \in \mathbb{N}$, all $i \in \mathcal{D}$, and all $t \in [0, \infty)$ we have

$$\begin{aligned} \mathbb{E} [H_t^N(i)] &= \mathbb{E} [Y_t^{N,t}(i)] \leq z_t^{N,t}(i) \leq \max \left\{ \mathbb{E} [Y_0^{N,t}(i)], \limsup_{s \rightarrow \infty} z_s^{N,t}(i) \right\} = \max \left\{ \mathbb{E} [Y_0^{N,t}(i)], c_1 \right\} \\ &\leq \mathbb{E} [Y_0^{N,t}(i)] + c_1 = \mathbb{E} \left[\sum_{j \in \mathcal{D}} m_{t\kappa_H^N}(i, j) H_0^N(j) \right] + c_1 \end{aligned} \quad (114)$$

This finishes the proof of Lemma 2.11. \square

Proof of Theorem 1.3. We will use stochastic averaging (see Theorem 2.1 in Kurtz [21]) to prove the result. So we first check that all conditions of the aforementioned theorem are fulfilled. Note that $E_1 = [0, 1]^\mathcal{D}$ and $E_2 = l_\sigma^1 \cap [0, \infty)^\mathcal{D}$ are complete separable metric spaces. Tychonoff's theorem implies that E_1 is compact. Since for all $N \in \mathbb{N}$ and all $t \in [0, \infty)$ the random variable F_{tN}^N takes values in the compact space E_1 , the compact containment condition holds for $\{(F_{tN}^N)_{t \in [0, \infty)} : N \in \mathbb{N}\}$. We will now use Lemma 2.10 to show for each $T \in [0, \infty)$ that the family $\{H_{tN}^N : t \in [0, T], N \in \mathbb{N}\}$ is relatively compact in E_2 . From Lemma 2.5 and the assumption $\sup_{N \in \mathbb{N}} \mathbb{E} [\| (H_0^N + P_0^N)^4 \|] < \infty$ we see that

$$\sup_{N \in \mathbb{N}} \sup_{t \in [0, \infty)} \mathbb{E} [\|H_{tN}^N\|_\sigma] < \infty. \quad (115)$$

Define $\mathcal{D}_0 := \emptyset$ and for all $n \in \mathbb{N}$ let $\mathcal{D}_n \subseteq \mathcal{D}$ be a set with $|\mathcal{D}_n| = \min\{n, |\mathcal{D}|\}$ and $\mathcal{D}_n \supseteq \mathcal{D}_{n-1}$. Define $c_1 := \frac{K}{2} \left(1 + \sqrt{1 + \frac{4\bar{\iota}_H}{K\lambda}} \right)$. From Lemma 2.11 with the assumption that $\sum_{i \in \mathcal{D}} \sup_{N \in \mathbb{N}} \sigma_i \mathbb{E} [H_0^N(i)] < \infty$ we get for all $T \in [0, \infty)$ that

$$\begin{aligned} \sum_{i \in \mathcal{D}} \sigma_i \sup_{N \in \mathbb{N}} \sup_{t \in [0, T]} \mathbb{E} [H_{tN}^N(i)] &\leq \sum_{i \in \mathcal{D}} \sigma_i \sup_{N \in \mathbb{N}} \sup_{t \in [0, T]} \left(\sum_{j \in \mathcal{D}} \sum_{n=0}^{\infty} e^{-tN\kappa_H^N} \frac{(tN\kappa_H^N)^n}{n!} m^n(i, j) \mathbb{E} [H_0^N(j)] + c_1 \right) \\ &\leq \sum_{j \in \mathcal{D}} \sum_{n=0}^{\infty} \left(\sum_{i \in \mathcal{D}} \sigma_i m^n(i, j) \right) \sup_{N \in \mathbb{N}} \sup_{t \in [0, TN\kappa_H^N]} e^{-t} \frac{t^n}{n!} \mathbb{E} [H_0^N(j)] + c_1 \sum_{i \in \mathcal{D}} \sigma_i \\ &\leq \sum_{j \in \mathcal{D}} \sum_{n=0}^{\infty} c^n \sigma_j \sup_{N \in \mathbb{N}} \frac{(TN\kappa_H^N)^n}{n!} \mathbb{E} [H_0^N(j)] + c_1 \|\mathbb{1}\|_\sigma \leq e^{cT} \sup_{M \in \mathbb{N}} M \kappa_H^M \sum_{j \in \mathcal{D}} \sigma_j \sup_{N \in \mathbb{N}} \mathbb{E} [H_0^N(j)] + c_1 \|\mathbb{1}\|_\sigma < \infty. \end{aligned} \quad (116)$$

Now we can use the dominated convergence theorem to obtain for all $T \in [0, \infty)$ that

$$\lim_{n \rightarrow \infty} \sup_{N \in \mathbb{N}} \sup_{t \in [0, T]} \sum_{k \in \mathcal{D} \setminus \mathcal{D}_n} \sigma_k \mathbb{E} [H_{tN}^N(k)] \leq \lim_{n \rightarrow \infty} \sum_{k \in \mathcal{D} \setminus \mathcal{D}_n} \sup_{N \in \mathbb{N}} \sup_{t \in [0, T]} \sigma_k \mathbb{E} [H_{tN}^N(k)] = 0. \quad (117)$$

Hence, for all $T \in [0, \infty)$ we can apply Lemma 2.10 to the family $\{H_{tN}^N : t \in [0, T], N \in \mathbb{N}\}$ and conclude that it is relatively compact in E_2 . Denote by $C_b(E_1, \mathbb{R})$ the set of bounded, continuous real-valued functions on E_1 and by $C_b^2(E_1, \mathbb{R})$ the set of all real-valued functions on E_1 that are twice continuously differentiable and bounded, with bounded first and second order partial derivatives. For $f \in C_b^2(E_1, \mathbb{R})$ let $c_f \in (0, \infty)$ be such that for all $x \in E_1$ and all $i \in \mathcal{D}$ we have $\left| \frac{df}{dx_i}(x) \right| + \left| \frac{d^2 f}{dx_i^2}(x) \right| \leq c_f$. Define

$$\text{Dom}(\mathcal{A}) := \{f \in C_b^2(E_1, \mathbb{R}) : f \text{ depends only on finitely many coordinates}\} \quad (118)$$

and for any $f \in \text{Dom}(\mathcal{A})$ denote by \mathcal{D}_f the finite set of coordinates that f depends on. Due to the Stone-Weierstrass theorem (e.g., Theorem 15.2 in Klenke [19]) we see that $\text{Dom}(\mathcal{A})$ is dense in $C_b(E_1, \mathbb{R})$ in the topology of uniform convergence. Denote by $C(E_1 \times E_2, \mathbb{R})$ the set of real-valued continuous functions on $E_1 \times E_2$ and define the operator $\mathcal{A}_1 : \text{Dom}(\mathcal{A}) \rightarrow C(E_1 \times E_2, \mathbb{R})$ for all $f \in \text{Dom}(\mathcal{A})$, all $x \in E_1$, and all $y \in E_2$ by

$$(\mathcal{A}_1 f)(x, y) := \sum_{i \in \mathcal{D}} \mathbb{1}_{y_i > 0} \left(\left[\kappa_H \sum_{j \in \mathcal{D}} \left(m(i, j) \frac{y_j}{y_i} (x_j - x_i) \right) - \alpha x_i (1 - x_i) \right] \frac{df}{dx_i}(x) + \frac{1}{2} \beta_H \frac{x_i(1-x_i)}{y_i} \frac{d^2 f}{dx_i^2}(x) \right). \quad (119)$$

For all $f \in \text{Dom}(\mathcal{A})$, all $N \in \mathbb{N}$, and all $t \in [0, \infty)$ define

$$\begin{aligned} \varepsilon_f^N(t) := & \frac{1}{N} \int_0^t (\mathcal{A}_1 f)(F_u^N, H_u^N) du - \sum_{i \in \mathcal{D}} \int_0^t \frac{df}{dx_i}(F_u^N) \left[\kappa_H^N \sum_{j \in \mathcal{D}} m(i, j) (F_u^N(j) - F_u^N(i)) \frac{H_u^N(j)}{H_u^N(i)} \right. \\ & \left. - \alpha^N F_u^N(i) (1 - F_u^N(i)) \right] + \frac{1}{2} \frac{d^2 f}{dx_i^2}(F_u^N) \frac{\beta_H F_u^N(i)(1-F_u^N(i))}{N H_u^N(i)} du. \end{aligned} \quad (120)$$

From Itô's lemma and Lemma 2.4 we get for all $f \in \text{Dom}(\mathcal{A})$, all $N \in \mathbb{N}$, and all $t \in [0, \infty)$ that \mathbb{P} -a.s.

$$\begin{aligned} f(F_t^N) - f(F_0^N) &= \sum_{i \in \mathcal{D}} \int_0^t \frac{df}{dx_i}(F_u^N) dF_u^N(i) + \frac{1}{2} \sum_{i, j \in \mathcal{D}} \int_0^t \left(\frac{d^2 f}{dx_i dx_j}(F_u^N) \right) d\langle F^N(i), F^N(j) \rangle_u \\ &= \sum_{i \in \mathcal{D}} \int_0^t \frac{df}{dx_i}(F_u^N) \left[\kappa_H^N \sum_{j \in \mathcal{D}} m(i, j) (F_u^N(j) - F_u^N(i)) \frac{H_u^N(j)}{H_u^N(i)} - \alpha^N F_u^N(i) (1 - F_u^N(i)) \right] \\ &\quad + \frac{1}{2} \frac{d^2 f}{dx_i^2}(F_u^N) \frac{\beta_H^N F_u^N(i)(1-F_u^N(i))}{H_u^N(i)} du + \sum_{i \in \mathcal{D}} \int_0^t \frac{df}{dx_i}(F_u^N) \sqrt{\frac{\beta_H^N F_u^N(i)(1-F_u^N(i))}{H_u^N(i)}} dW_u^{F, N}(i). \end{aligned} \quad (121)$$

Hence, we get for all $f \in \text{Dom}(\mathcal{A})$, all $N \in \mathbb{N}$, and all $t \in [0, \infty)$ that \mathbb{P} -a.s.

$$\begin{aligned} f(F_{tN}^N) - \int_0^t (\mathcal{A}_1 f)(F_{uN}^N, H_{uN}^N) du + \varepsilon_f^N(tN) \\ = f(F_0^N) + \sum_{i \in \mathcal{D}} \int_0^t \frac{df}{dx_i}(F_{uN}^N) \sqrt{\frac{\beta_H^N F_{uN}^N(i)(1-F_{uN}^N(i))}{H_{uN}^N(i)}} dW_{uN}^{F, N}(i). \end{aligned} \quad (122)$$

From Tonelli's theorem and Lemma 2.6 we obtain for all $f \in \text{Dom}(\mathcal{A})$, all $N \in \mathbb{N}$, and all $t \in [0, \infty)$ that

$$\begin{aligned} \mathbb{E} \left[\int_0^t \left(\sum_{i \in \mathcal{D}} \frac{df}{dx_i}(F_u^N) \sqrt{\frac{\beta_H^N F_u^N(i)(1-F_u^N(i))}{H_u^N(i)}} \right)^2 du \right] &\leq t |\mathcal{D}_f| c_f^2 \bar{\beta}_H \max_{i \in \mathcal{D}_f} \sup_{M \in \mathbb{N}} \sup_{u \in [0, \infty)} \mathbb{E} \left[\frac{1}{H_u^M(i)} \right] \\ &\leq t |\mathcal{D}_f| c_f^2 \bar{\beta}_H \max_{i \in \mathcal{D}_f} \frac{1}{\sigma_i} \sup_{M \in \mathbb{N}} \sup_{u \in [0, \infty)} \mathbb{E} \left[\left\| \frac{1}{H_u^M} \right\|_\sigma \right] < \infty. \end{aligned} \quad (123)$$

Thus for all $f \in \text{Dom}(\mathcal{A})$, all $N \in \mathbb{N}$, and all $t \in [0, \infty)$ the left-hand side of (122) is a martingale. Next, for all

$f \in \text{Dom}(\mathcal{A})$ and all $T \in [0, \infty)$ it holds that

$$\begin{aligned}
& \sup_{N \in \mathbb{N}} \mathbb{E} \left[\int_0^T |(\mathcal{A}_1 f)(F_{tN}^N, H_{tN}^N)|^{\frac{4}{3}} dt \right] \\
&= \sup_{N \in \mathbb{N}} \mathbb{E} \left[\int_0^T \left| \sum_{i \in \mathcal{D}_f} \left(\kappa_H \sum_{j \in \mathcal{D}} \left(m(i, j) \frac{H_{tN}^N(j)}{H_{tN}^N(i)} (F_{tN}^N(j) - F_{tN}^N(i)) \right) \right. \right. \right. \\
&\quad \left. \left. - \alpha F_{tN}^N(i) (1 - F_{tN}^N(i)) \right) \frac{df}{dx_i}(F_{tN}^N) + \frac{1}{2} \sum_{i \in \mathcal{D}_f} \beta_H \frac{F_{tN}^N(i)(1 - F_{tN}^N(i))}{H_{tN}^N(i)} \frac{d^2 f}{dx_i^2}(F_{tN}^N) \right|^{\frac{4}{3}} dt \right] \\
&\leq \sup_{N \in \mathbb{N}} \mathbb{E} \left[\int_0^T \left(\sum_{i \in \mathcal{D}_f} \left(\left| \kappa_H \sum_{j \in \mathcal{D}} \left(m(i, j) \frac{H_{tN}^N(j)}{H_{tN}^N(i)} c_f \right) \right| + |\alpha c_f| + \left| \frac{1}{2} \beta_H \frac{1}{H_{tN}^N(i)} c_f \right| \right) \right)^{\frac{4}{3}} dt \right].
\end{aligned} \tag{124}$$

Using Young's inequality and Jensen's inequality we get for all $f \in \text{Dom}(\mathcal{A})$ and all $T \in [0, \infty)$ that

$$\begin{aligned}
& \sup_{N \in \mathbb{N}} \mathbb{E} \left[\int_0^T |(\mathcal{A}_1 f)(F_{tN}^N, H_{tN}^N)|^{\frac{4}{3}} dt \right] \\
&\leq \sup_{N \in \mathbb{N}} \mathbb{E} \left[\int_0^T \left(\sum_{i \in \mathcal{D}_f} \left(\frac{2}{3} \left(\kappa_H c_f \frac{1}{H_{tN}^N(i)} \right)^{\frac{3}{2}} + \frac{1}{3} \left(\sum_{j \in \mathcal{D}} m(i, j) H_{tN}^N(j) \right)^3 + \alpha c_f + \frac{1}{2} \beta_H \frac{1}{H_{tN}^N(i)} c_f \right) \right)^{\frac{4}{3}} dt \right] \\
&\leq \sup_{N \in \mathbb{N}} \mathbb{E} \left[\int_0^T \frac{(4|\mathcal{D}_f|)^{\frac{1}{3}}}{\min_{k \in \mathcal{D}_f} \{\sigma_k\}} \sum_{i \in \mathcal{D}_f} \sigma_i \left(\left(\frac{2}{3} \right)^{\frac{4}{3}} \left(\kappa_H c_f \frac{1}{H_{tN}^N(i)} \right)^2 + \left(\frac{1}{3} \right)^{\frac{4}{3}} \left(\sum_{j \in \mathcal{D}} m(i, j) H_{tN}^N(j) \right)^4 \right. \right. \\
&\quad \left. \left. + \left(\alpha c_f \right)^{\frac{4}{3}} + \left(\frac{1}{2} \beta_H \frac{1}{H_{tN}^N(i)} c_f \right)^{\frac{4}{3}} \right) dt \right].
\end{aligned} \tag{125}$$

Using Lemma 2.4, Tonelli's theorem, and Lemmas 2.5 and 2.6 we obtain for all $f \in \text{Dom}(\mathcal{A})$ and all $T \in [0, \infty)$ that

$$\begin{aligned}
& \sup_{N \in \mathbb{N}} \mathbb{E} \left[\int_0^T |(\mathcal{A}_1 f)(F_{tN}^N, H_{tN}^N)|^{\frac{4}{3}} dt \right] \\
&\leq \sup_{N \in \mathbb{N}} \frac{(4|\mathcal{D}_f|)^{\frac{2}{3}}}{\min_{k \in \mathcal{D}_f} \{\sigma_k\}} \int_0^T \left(\frac{4}{9} \right)^{\frac{2}{3}} (\kappa_H c_f)^2 \mathbb{E} \left[\left\| \frac{1}{(H_{tN}^N)^2} \right\|_{\sigma} \right] + \left(\frac{1}{9} \right)^{\frac{2}{3}} c \mathbb{E} \left[\left\| (H_{tN}^N)^4 \right\|_{\sigma} + (\alpha c_f)^{\frac{4}{3}} \|\mathbb{1}\|_{\sigma} \right] \\
&\quad + \left(\frac{1}{4} \right)^{\frac{2}{3}} (\beta_H c_f)^{\frac{4}{3}} \mathbb{E} \left[\left\| \left(\frac{1}{H_{tN}^N} \right)^{\frac{4}{3}} \right\|_{\sigma} \right] dt < \infty.
\end{aligned} \tag{126}$$

Furthermore, for all $f \in \text{Dom}(\mathcal{A})$, all $N \in \mathbb{N}$, and all $T \in [0, \infty)$ we have that

$$\begin{aligned}
& \mathbb{E} \left[\sup_{t \in [0, T]} |\varepsilon_f^N(tN)| \right] \\
&= \mathbb{E} \left[\sup_{t \in [0, T]} \left| \sum_{i \in \mathcal{D}_f} \int_0^t \frac{df}{dx_i}(F_{uN}^N) \left[(\kappa_H - N \kappa_H^N) \sum_{j \in \mathcal{D}} m(i, j) (F_{uN}^N(j) - F_{uN}^N(i)) \frac{H_{uN}^N(j)}{H_{uN}^N(i)} \right. \right. \right. \\
&\quad \left. \left. + (\alpha - N \alpha^N) F_{uN}^N(i) (1 - F_{uN}^N(i)) \right] + \frac{1}{2} \frac{d^2 f}{dx_i^2}(F_{uN}^N) (\beta_H - N \beta_H^N) \frac{F_{uN}^N(i)(1 - F_{uN}^N(i))}{H_{uN}^N(i)} du \right| \right] \\
&\leq \mathbb{E} \left[\int_0^T \sum_{i \in \mathcal{D}_f} c_f \left(|\kappa_H - N \kappa_H^N| \sum_{j \in \mathcal{D}} m(i, j) \frac{H_{uN}^N(j)}{H_{uN}^N(i)} + |\alpha - N \alpha^N| + \frac{1}{2} |\beta_H - N \beta_H^N| \frac{1}{H_{uN}^N(i)} \right) du \right].
\end{aligned} \tag{127}$$

Using Young's inequality, Lemma 2.4, and Tonelli's theorem we get for all $f \in \text{Dom}(\mathcal{A})$, all $N \in \mathbb{N}$, and all $T \in [0, \infty)$ that

$$\begin{aligned} \mathbb{E} \left[\sup_{t \in [0, T]} |\varepsilon_f^N(tN)| \right] &\leq \mathbb{E} \left[\int_0^T \sum_{i \in \mathcal{D}_f} \frac{\sigma_i c_f}{\min_{k \in \mathcal{D}_f} \{\sigma_k\}} \left(\frac{|\kappa_H - N\kappa_H^N|}{2} \left(\left(\frac{1}{H_{uN}^N(i)} \right)^2 + \left(\sum_{j \in \mathcal{D}} m(i, j) H_{uN}^N(j) \right)^2 \right) \right. \right. \\ &\quad \left. \left. + |\alpha - N\alpha^N| + \frac{|\beta_H - N\beta_H^N|}{2} \frac{1}{H_{uN}^N(i)} \right) du \right] \\ &\leq \frac{c_f}{\min_{k \in \mathcal{D}_f} \{\sigma_k\}} \int_0^T \frac{|\kappa_H - N\kappa_H^N|}{2} \left(\mathbb{E} \left[\left\| \frac{1}{(H_{uN}^N)^2} \right\|_\sigma \right] + c \mathbb{E} \left[\left\| (H_{uN}^N)^2 \right\|_\sigma \right] \right) \\ &\quad + |\alpha - N\alpha^N| \|\mathbb{1}\|_\sigma + \frac{|\beta_H - N\beta_H^N|}{2} \mathbb{E} \left[\left\| \frac{1}{H_{uN}^N} \right\|_\sigma \right] du. \end{aligned} \quad (128)$$

Hence, from Lemmas 2.5 and 2.6 we see for all $f \in \text{Dom}(\mathcal{A})$ and all $T \in [0, \infty)$ that

$$\begin{aligned} 0 &\leq \lim_{N \rightarrow \infty} \mathbb{E} \left[\sup_{t \in [0, T]} |\varepsilon_f^N(tN)| \right] \\ &\leq \lim_{N \rightarrow \infty} \frac{T c_f}{\min_{k \in \mathcal{D}_f} \{\sigma_k\}} \left(\frac{|\kappa_H - N\kappa_H^N|}{2} \sup_{M \in \mathbb{N}} \sup_{t \in [0, \infty)} \left(\mathbb{E} \left[\left\| \frac{1}{(H_t^M)^2} \right\|_\sigma \right] + c \mathbb{E} \left[\left\| (H_t^M)^2 \right\|_\sigma \right] \right) + |\alpha - N\alpha^N| \|\mathbb{1}\|_\sigma \right. \\ &\quad \left. + \frac{|\beta_H - N\beta_H^N|}{2} \sup_{M \in \mathbb{N}} \sup_{t \in [0, \infty)} \mathbb{E} \left[\left\| \frac{1}{H_t^M} \right\|_\sigma \right] \right) = 0. \end{aligned} \quad (129)$$

Define the set $\mathcal{R} := \left\{ \times_{i \in \mathcal{D}} B_i : (B_i)_{i \in \mathcal{D}} \subseteq \mathcal{B}([0, \infty)^\mathcal{D}), B_i = [0, \infty) \text{ for all but finitely many } i \in \mathcal{D} \right\}$. For all $N \in \mathbb{N}$, all $t \in [0, \infty)$, and all $B \in \mathcal{R}$ define the measure-valued random variables

$$\Lambda^N([0, t] \times B) := \int_0^t \mathbb{1}_B(H_{uN}^N) du = \int_0^t \prod_{i \in \mathcal{D}} \mathbb{1}_{B_i}(H_{uN}^N(i)) du, \quad (130)$$

Due to Carathéodory's theorem (see e.g., Theorem 1.41 in Klenke [19]) there is a unique extension of this pre-measure to a measure on $[0, t] \times E_2$, which we will denote by the same name. Define the space $\ell(E_2) := \{\mu : \mu \text{ is a measure on } [0, \infty) \times E_2 \text{ such that for all } t \in [0, \infty) \text{ it holds that } \mu([0, t] \times E_2) = t\}$ and the space $D([0, \infty)) := \{f : [0, \infty) \rightarrow E_1 | f \text{ is càdlàg}\}$. Having checked all assumptions, we can now apply Theorem 2.1 from Kurtz [21] and conclude that the sequence $\{((F_{tN}^N)_{t \in [0, \infty)}, \Lambda^N) : N \in \mathbb{N}\}$ is relatively compact in $D([0, \infty)) \times \ell(E_2)$. Let (F, Λ) be a $D([0, \infty)) \times \ell(E_2)$ -valued random variable and let $(N_k)_{k \in \mathbb{N}} \subseteq \mathbb{N}$ be an increasing sequence such that $\lim_{k \rightarrow \infty} ((F_{tN_k}^{N_k})_{t \in [0, \infty)}, \Lambda^{N_k}) = (F, \Lambda)$. Due to Skorohod's representation theorem (see Theorem 3.1.8 of Ethier and Kurtz [7]) we can assume without loss of generality and for ease of notation that (F, Λ) acts on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Using Hölder's inequality and Theorem 1.2 we see for all $t \in [0, \infty)$ that

$$\begin{aligned} 0 &\leq \lim_{N \rightarrow \infty} \int_0^t \mathbb{E} \left[\left\| H_{uN}^N - (h_\infty(F_{uN}^N(i)))_{i \in \mathcal{D}} \right\|_\sigma \right] du = \lim_{N \rightarrow \infty} \int_0^t \mathbb{E} \left[\left| \sum_{i \in \mathcal{D}} \sigma_i |H_{uN}^N(i) - h_\infty(F_{uN}^N(i))| \right| \right] du \\ &\leq \lim_{N \rightarrow \infty} \sqrt{\int_0^t \mathbb{E} \left[\sum_{i \in \mathcal{D}} \sigma_i (H_{uN}^N(i) - h_\infty(F_{uN}^N(i)))^2 \right] du} \sqrt{t \sum_{k \in \mathcal{D}} \sigma_k} = 0. \end{aligned} \quad (131)$$

For any bounded Lipschitz continuous function $f : l_\sigma^1 \rightarrow \mathbb{R}$, with Lipschitz constant \bar{c}_f , and all $t \in [0, \infty)$, applying (131), we then have

$$\begin{aligned} 0 &\leq \mathbb{E} \left[\int_0^t \int_{E_2} f(y) \Lambda(du \times dy) - \int_0^t f(h_\infty(F_u)) du \right] \\ &= \lim_{k \rightarrow \infty} \mathbb{E} \left[\int_0^t f(H_{uN_k}^{N_k}) du - \int_0^t f(h_\infty(F_{uN_k}^{N_k})) du \right] \leq \bar{c}_f \lim_{k \rightarrow \infty} \mathbb{E} \left[\int_0^t \|H_{uN_k}^{N_k} - h_\infty(F_{uN_k}^{N_k})\| du \right] = 0. \end{aligned} \quad (132)$$

Define the operator $\mathcal{A}_2 : \text{Dom}(\mathcal{A}) \rightarrow C(E_1, \mathbb{R})$ for all $f \in \text{Dom}(\mathcal{A})$ and all $x \in E_1$ by

$$\begin{aligned} (\mathcal{A}_2 f)(x) &:= \sum_{i \in \mathcal{D}} \left(\kappa_H \sum_{j \in \mathcal{D}} \left(m(i, j) \frac{a-x_i}{a-x_j} (x_j - x_i) \right) - \alpha x_i (1 - x_i) \right) \frac{df}{dx_i}(x) \\ &\quad + \frac{1}{2} \sum_{i \in \mathcal{D}} \beta_H b(a - x_i) x_i (1 - x_i) \frac{d^2 f}{dx_i^2}(x). \end{aligned} \quad (133)$$

For all $t \in [0, \infty)$, all $f \in \text{Dom}(\mathcal{A})$, and all $x \in E_1$ we have \mathbb{P} -a.s.

$$\begin{aligned} &\int_0^t \int_{E_2} (\mathcal{A}_1 f)(F_s, y) \Lambda(ds \times dy) \\ &= \int_0^t \int_{E_2} \left[\sum_{i \in \mathcal{D}} \left(\kappa_H \sum_{j \in \mathcal{D}} \left(m(i, j) \frac{y_i}{y_i} (F_s(j) - F_s(i)) \right) - \alpha F_s(i) (1 - F_s(i)) \right) \frac{df}{dx_i}(F_s) \right. \\ &\quad \left. + \frac{1}{2} \sum_{i \in \mathcal{D}} \beta_H \frac{F_s(i)(1-F_s(i))}{y_i} \frac{d^2 f}{dx_i^2}(F_s) \right] \prod_{i \in \mathcal{D}} \mathbb{1}_{y_i}(h_\infty(F_s(i))) dy ds = \int_0^t (\mathcal{A}_2 f)(F_s) ds. \end{aligned} \quad (134)$$

Applying Theorem 2.1 of Kurtz [21] together with (134), we see for each $f \in \text{Dom}(\mathcal{A})$ that

$$(f(F_t) - \int_0^t (\mathcal{A}_2 f)(F_s) ds)_{t \in [0, \infty)} \quad (135)$$

is a martingale. Hence, F is a solution to (5). Note that for all $z_1, z_2 \in [0, 1]$ we have that $\frac{a-z_1}{a-z_2}(z_2 - z_1) = (a - z_1)(\frac{a-z_1}{a-z_2} - 1)$. Using this and (3) we then have for any subset $\mathcal{S} \subseteq \mathcal{D}$ and any $x, y \in E_2$ that

$$\begin{aligned} &\sum_{i \in \mathcal{S}} \sigma_i \mathbb{1}_{x_i \geq y_i} \left(\kappa_H \sum_{j \in \mathcal{D}} m(i, j) \left(\frac{(a-x_i)^2}{a-x_j} - (a-x_i) - \frac{(a-y_i)^2}{a-y_j} + (a-y_i) \right) - \alpha(x_i(1-x_i) - y_i(1-y_i)) \right) \\ &= \sum_{i \in \mathcal{S}} \sigma_i \mathbb{1}_{x_i \geq y_i} \left(\kappa_H \sum_{j \in \mathcal{D}} m(i, j) \left((x_i - y_i) + ((a-x_i)^2 - (a-y_i)^2) \frac{1}{a-x_j} - (a-y_i)^2 \left(\frac{1}{a-y_j} - \frac{1}{a-x_j} \right) \right) \right. \\ &\quad \left. + \alpha(-(x_i - y_i) + x_i^2 - y_i^2) \right) \\ &\leq \sum_{i \in \mathcal{S}} \sigma_i \left(\kappa_H \sum_{j \in \mathcal{D}} m(i, j) \mathbb{1}_{x_j \geq y_j} (a-y_i)^2 \left(\frac{1}{a-x_j} - \frac{1}{a-y_j} \right) \right) + \sum_{i \in \mathcal{S}} \sigma_i (\kappa_H + 2\alpha) \mathbb{1}_{x_i \geq y_i} (x_i - y_i) \\ &\leq \sum_{i \in \mathcal{S}} \sigma_i \left(\kappa_H \sum_{j \in \mathcal{D}} m(i, j) \mathbb{1}_{x_j \geq y_j} \frac{a^2}{(a-1)^2} (x_j - y_j) \right) + \sum_{i \in \mathcal{S}} \sigma_i (\kappa_H + 2\alpha) \mathbb{1}_{x_i \geq y_i} (x_i - y_i) \\ &\leq \sum_{i \in \mathcal{S}} \sigma_i c \kappa_H \mathbb{1}_{x_i \geq y_i} \frac{a^2(x_i - y_i)}{(a-1)^2} + \sum_{i \in \mathcal{S}} \sigma_i (\kappa_H + 2\alpha) \mathbb{1}_{x_i \geq y_i} (x_i - y_i) = \sum_{i \in \mathcal{S}} \sigma_i \left(\frac{c \kappa_H a^2}{(a-1)^2} + \kappa_H + 2\alpha \right) (x_i - y_i)^+. \end{aligned} \quad (136)$$

This implies that equation (26) of Hutzenthaler and Wakolbinger [17] is fulfilled. Together with the assumptions on m in Assumption 1.1 we now infer, analogous to Proposition 2.1 of Hutzenthaler and Wakolbinger [17], that the system (5) has a unique strong solution with a.s. continuous paths. We conclude that any limit point of $\{(F_{tN}^N)_{t \in [0, \infty)} : N \in \mathbb{N}\}$ solves (5). Combining this with the fact that $\{(F_{tN}^N)_{t \in [0, \infty)} : N \in \mathbb{N}\}$ is relatively compact we obtain $(F_{tN}^N)_{t \in [0, \infty)} \Rightarrow (X_t)_{t \in [0, \infty)}$, as $N \rightarrow \infty$. This finishes the proof of Theorem 1.3. \square

3 McKean-Vlasov limit

In this section we investigate convergence of a sequence of exchangeable systems of stochastic differential equations.

3.1 Setting

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $I \subset [0, \infty)$ be an interval of length $|I| \in (0, \infty]$ which is either of the form $[0, |I|]$ if $|I| < \infty$ or of the form $[0, \infty)$ if $|I| = \infty$, let $A \subseteq \mathbb{R}$ be a convex set, and let $\psi: I \rightarrow A$, $\xi: A \times I \rightarrow \mathbb{R}$, and $\sigma^2: I \rightarrow [0, \infty)$ be functions. The function $\sigma^2: I \rightarrow [0, \infty)$ is locally Lipschitz continuous in I and satisfies $\sigma^2(0) = 0$ and if $|I| < \infty$, then $\sigma^2(|I|) = 0$. Furthermore, the function σ^2 is strictly positive on $(0, |I|)$. There exists a constant $L \in (0, \infty)$ such that σ^2 satisfies the growth condition that for all $y \in I$ we have $\sigma^2(y) \leq L(y + y^2)$ and such that ξ satisfies for all $(u, x), (v, y) \in A \times I$ that

$$\mathbb{1}_{x \geq y} (\xi(u, x) - \xi(v, y)) \leq L(u - v)^+ + L(x - y)^+. \quad (137)$$

The function $\psi: I \rightarrow [0, \infty)$ satisfies for all $x, y \in I$ that $|\psi(x) - \psi(y)| \leq L|x - y|$. Let $W(i): [0, \infty) \times \Omega \rightarrow \mathbb{R}$, $i \in \mathbb{N}$, be independent Brownian motions with continuous sample paths. For all $D \in \mathbb{N}$ let $X^D: [0, \infty) \times \{1, \dots, D\} \times \Omega \rightarrow I$ be an adapted stochastic process with continuous sample paths that for all $t \in [0, \infty)$ and all $i \in \{1, \dots, D\}$ \mathbb{P} -a.s. satisfies

$$X_t^D(i) = X_0^D(i) + \int_0^t \xi\left(\frac{1}{D} \sum_{j \in \{1, \dots, D\}} \psi(X_s^D(j)), X_s^D(i)\right) ds + \int_0^t \sqrt{\sigma^2(X_s^D(i))} dW_s(i). \quad (138)$$

Let $M: [0, \infty) \times \Omega \rightarrow I$ be an adapted stochastic process with continuous sample paths that for all $t \in [0, \infty)$ \mathbb{P} -a.s. satisfies

$$M_t = M_0 + \int_0^t \xi(\mathbb{E}[\psi(M_s)], M_s) ds + \int_0^t \sqrt{\sigma^2(M_s)} dW_s(1). \quad (139)$$

3.2 McKean-Vlasov limit

The following proposition, Proposition 3.1, partly generalizes Proposition 4.29 in Hutzenthaler [16] where ξ depends linearly on its first argument.

Proposition 3.1. *Assume the setting of Section 3.1, let M_0 be an I -valued random variable, for every $D \in \mathbb{N}$ let $(X_0^D(j))_{j \in \{1, \dots, D\}}$ be exchangeable and integrable random variables with values in I . Then, there exists a unique solution M of (139) and for all $D \in \mathbb{N}$ and all $t \in [0, \infty)$ we have that*

$$\sqrt{D} \mathbb{E}[|X_t^D(1) - M_t|] \leq e^{(L^2 + L + L_\mu)t} \left(\sqrt{D} \mathbb{E}[|X_0^D(1) - M_0|] + L \int_0^t \left(\text{Var}(\psi(M_s)) \right)^{\frac{1}{2}} ds \right). \quad (140)$$

Proof. Existence of a weak solution is straightforward using a tightness argument. Next we show pathwise uniqueness for the SDE (139). Let $M, \bar{M}: [0, \infty) \times \Omega \rightarrow I$ be two solutions of the SDE (139). Then our assumptions and a standard Yamada-Watanabe argument (cf., e.g., Theorem 1 in Yamada and Watanabe [42]) shows for all $t \in [0, \infty)$ that \mathbb{P} -a.s.

$$|M_t - \bar{M}_t| = |M_0 - \bar{M}_0| + \int_0^t \text{sgn}(M_s - \bar{M}_s) d(M_s - \bar{M}_s). \quad (141)$$

Let $(\tau_l)_{l \in \mathbb{N}}$ be a localizing sequence for the local martingale $(\int_0^t \text{sgn}(M_s - \bar{M}_s)(\sigma^2(M_s) - \sigma^2(\bar{M}_s)) dW_s)_{t \in [0, \infty)}$. Then Fatou's Lemma and our assumptions imply for all $t \in [0, \infty)$ that

$$\begin{aligned} \mathbb{E}[|M_t - \bar{M}_t|] &\leq \lim_{l \rightarrow \infty} \mathbb{E}[|M_{t \wedge \tau_l} - \bar{M}_{t \wedge \tau_l}|] \\ &\leq \mathbb{E}[|M_0 - \bar{M}_0|] + \mathbb{E}\left[\int_0^t \text{sgn}(M_s - \bar{M}_s) (\xi(\mathbb{E}[\psi(M_s)], M_s) - \xi(\mathbb{E}[\psi(\bar{M}_s)], \bar{M}_s)) ds\right] \\ &\leq \mathbb{E}[|M_0 - \bar{M}_0|] + L \int_0^t |\mathbb{E}[\psi(M_s)] - \mathbb{E}[\psi(\bar{M}_s)]| + \mathbb{E}[|M_s - \bar{M}_s|] ds \\ &\leq \mathbb{E}[|M_0 - \bar{M}_0|] + (L + 1)^2 \int_0^t \mathbb{E}[|M_s - \bar{M}_s|] ds. \end{aligned} \quad (142)$$

This together with Gronwall's lemma implies pathwise uniqueness for the SDE (139). Therefore, the theorem of Yamada and Watanabe (see Yamada and Watanabe [42]) implies that the SDE (139) is exact. The rest of the proof is analogous to the proof of Proposition 4.29 in Hutzenthaler [16] and we omit it here. \square

3.3 Application to altruistic defense in structured populations

In this section we verify the applicability of Proposition 3.1 to the case of altruistic defense in structured populations.

Lemma 3.2. *Let $\alpha, \beta, \kappa \in (0, \infty)$ and $a \in (1, \infty)$, let $I = [0, 1]$ and define the function $\sigma^2: I \rightarrow [0, \infty)$ by $I \ni x \mapsto \sigma^2(x) := \beta(a-x)x(1-x)$, the function $\psi: I \rightarrow [0, \infty)$ by $I \ni x \mapsto \psi(x) := \frac{1}{a-x}$, and the function $\xi: [0, \infty) \times I \rightarrow \mathbb{R}$ by $[0, \infty) \times I \ni (u, x) \mapsto \xi(u, x) := \kappa(a-x)((a-x)u-1) - \alpha x(1-x)$. Then the interval I and the functions σ^2 , ψ , and ξ satisfy the setting of Section 3.1 with $L = \max\{\beta a, \kappa a^2, \kappa + \alpha, \frac{1}{(a-1)^2}\}$.*

Proof. For all $(u, x), (v, y) \in [0, \infty) \times [0, 1]$ it holds that

$$\begin{aligned} & \mathbb{1}_{x \geq y} (\xi(u, x) - \xi(v, y)) \\ &= \mathbb{1}_{x \geq y} (\kappa(a-x)((a-x)u-1) - \kappa(a-y)((a-y)v-1) - \alpha x(1-x) + \alpha y(1-y)) \\ &= \mathbb{1}_{x \geq y} (\kappa[(a-x)^2 u - (a-x) - (a-y)^2 v + (a-y)] - \alpha(1-(x+y))(x-y)) \\ &= \mathbb{1}_{x \geq y} (\kappa[(x-y) + ((a-x)^2 - (a-y)^2)u - (a-y)^2(v-u)] - \alpha(1-(x+y))(x-y)) \\ &\leq (\kappa + \alpha)(x-y)^+ + \kappa a^2(u-v)^+ \leq L(x-y)^+ + L(u-v)^+. \end{aligned} \quad (143)$$

Moreover, for all $x, y \in I$ it holds that $\sigma^2(x) = \beta(a-x)x(1-x) \leq \beta a x \leq L(x+x^2)$ and that

$$|\psi(x) - \psi(y)| = \left| \frac{1}{a-x} - \frac{1}{a-y} \right| = \left| \int_y^x \frac{1}{(a-z)^2} dz \right| \leq \frac{1}{(a-1)^2} |x-y| \leq L|x-y|. \quad (144)$$

This finishes the proof of Lemma 3.2. \square

4 Long-term behavior of the average altruist frequency

4.1 Setting

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $\kappa, \alpha, \beta \in (0, \infty)$, $a \in (1, \infty)$, $c \in (0, 1)$, let $W: [0, \infty) \times \Omega \rightarrow \mathbb{R}$ be a Brownian motion with continuous sample paths, let $Z: [0, \infty) \times \Omega \rightarrow [0, 1]$ be an adapted process with continuous sample paths that for all $t \in [0, \infty)$ satisfies \mathbb{P} -a.s.

$$Z_t = Z_0 + \int_0^t (\kappa(a-Z_s)((a-Z_s)\mathbb{E}[\frac{1}{a-Z_s}] - 1) - \alpha Z_s(1-Z_s)) ds + \int_0^t \sqrt{\beta(a-Z_s)Z_s(1-Z_s)} dW_s. \quad (145)$$

Moreover, for all $\theta \in (\frac{1}{a}, \frac{1}{a-1})$ let $Z^\theta: [0, \infty) \times \Omega \rightarrow [0, 1]$ be an adapted process with continuous sample paths that for all $t \in [0, \infty)$ satisfies \mathbb{P} -a.s.

$$Z_t^\theta = Z_0^\theta + \int_0^t (\kappa(a-Z_s^\theta)((a-Z_s^\theta)\theta - 1) - \alpha Z_s^\theta(1-Z_s^\theta)) ds + \int_0^t \sqrt{\beta(a-Z_s^\theta)Z_s^\theta(1-Z_s^\theta)} dW_s. \quad (146)$$

For all $\theta \in (\frac{1}{a}, \frac{1}{a-1})$ and all $z \in [0, 1]$ define

$$\begin{aligned} m_\theta(z) &:= \beta c^{\frac{2\kappa}{\beta}(a\theta-1)} (1-c)^{\frac{2\kappa}{\beta}(1-\theta(a-1))} (a-c)^{\frac{2\alpha}{\beta}} \frac{1}{\beta(a-z)z(1-z)} \exp\left(\int_c^z 2 \frac{\kappa(a-y)((a-y)\theta-1) - \alpha y(1-y)}{\beta(a-y)y(1-y)} dy\right) \\ &= z^{\frac{2\kappa}{\beta}(a\theta-1)-1} (1-z)^{\frac{2\kappa}{\beta}(1-\theta(a-1))-1} (a-z)^{\frac{2\alpha}{\beta}-1}. \end{aligned} \quad (147)$$

Note that this defines the speed density (see p. 95 in Karlin and Taylor [18]) for (146). Furthermore, note that for all $\theta \in (\frac{1}{a}, \frac{1}{a-1})$ it holds that

$$\int_0^1 m_\theta(z) dz < \infty. \quad (148)$$

For all $\theta \in (\frac{1}{a}, \frac{1}{a-1})$ define $c_\theta := \int_0^1 m_\theta(z) dz$, for all $x \in \{0, 1\}$ denote by δ_x the Dirac measure on $[0, 1]$, and for all $\theta \in [\frac{1}{a}, \frac{1}{a-1}]$ define the mapping $\Psi_\theta : \mathcal{B}([0, 1]) \rightarrow [0, 1]$ by

$$\mathcal{B}([0, 1]) \ni A \mapsto \Psi_\theta(A) := \begin{cases} \delta_0(A), & \text{if } \theta = \frac{1}{a}, \\ \delta_1(A), & \text{if } \theta = \frac{1}{a-1}, \\ \int_A \frac{1}{c_\theta} m_\theta(z) dz, & \text{if } \theta \in (\frac{1}{a}, \frac{1}{a-1}). \end{cases} \quad (149)$$

4.2 Results for the equilibrium distribution

Assume the setting of Section 4.1. Existence and uniqueness of the solution of (145) follow from Proposition 3.1. When $\theta \in (\frac{1}{a}, \frac{1}{a-1})$ we have that Ψ_θ defines a probability distribution by (148), and we can apply Theorem V.54.5 of Rogers and Williams [34] to conclude that it is the unique equilibrium distribution for (146). The proof of the following lemma, Lemma 4.1, is clear and therefore omitted.

Lemma 4.1. *Assume the setting of Section 4.1. A probability measure $\Phi : \mathcal{B}([0, 1]) \rightarrow [0, 1]$ is an equilibrium distribution of the dynamics (145) if and only if there exists a $\theta \in [\frac{1}{a}, \frac{1}{a-1}]$ such that $\Phi = \Psi_\theta$.*

Lemma 4.2. *Assume the setting of Section 4.1 and let $\theta \in (\frac{1}{a}, \frac{1}{a-1})$. Then we have*

$$\int_0^1 \frac{1}{a-z} \Psi_\theta(dz) \begin{cases} < \theta, & \text{if } \alpha > \beta, \\ = \theta, & \text{if } \alpha = \beta, \\ > \theta, & \text{if } \alpha < \beta. \end{cases} \quad (150)$$

Proof. Define $u := \frac{2\kappa}{\beta}(a\theta - 1)$ and $v := \frac{2\kappa}{\beta}(1 - \theta(a - 1))$ and note that $u, v \in (0, \infty)$. Let $\Gamma : (0, \infty) \rightarrow (0, \infty)$ be the Gamma function, i.e., for all $x \in (0, \infty)$ let $\Gamma(x) := \int_0^\infty z^{x-1} e^{-z} dz$. It is well-known that for all $x \in (0, \infty)$ the Gamma function satisfies $\Gamma(x+1) = x\Gamma(x)$ and that for all $x, y \in (0, \infty)$ it holds that $\int_0^1 z^{x-1} (1-z)^{y-1} dz = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$. Thus, we obtain

$$\begin{aligned} & \int_0^1 z^{u-1} (1-z)^{v-1} (a-z) \left(\frac{1}{a-z} - \theta \right) dz \\ &= \int_0^1 z^{u-1} (1-z)^{v-1} dz - a\theta \int_0^1 z^{u-1} (1-z)^{v-1} dz + \theta \int_0^1 z^u (1-z)^{v-1} dz \\ &= \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)} - a\theta \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)} + \theta \frac{\Gamma(u+1)\Gamma(v)}{\Gamma(u+v+1)} = \left((1-a\theta) \frac{(u+v)\Gamma(u)\Gamma(v)}{\Gamma(u+v+1)} + \theta \frac{u\Gamma(u)\Gamma(v)}{\Gamma(u+v+1)} \right) \\ &= (u(1-a\theta) + \theta + v(1-a\theta)) \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v+1)} = \frac{2\kappa}{\beta} ((a\theta - 1)(1 - \theta(a - 1)) + (1 - \theta(a - 1))(1 - a\theta)) \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v+1)} \\ &= \left(\frac{2\kappa}{\beta} (1 - \theta(a - 1))(a\theta - 1 + 1 - a\theta) \right) \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v+1)} = 0. \end{aligned} \quad (151)$$

First, consider the case $\alpha = \beta$. Using (151) we see that

$$\begin{aligned} \int_0^1 \frac{1}{a-z} \Psi_\theta(dz) - \theta &= \int_0^1 c_\theta z^{\frac{2\kappa}{\beta}(a\theta-1)-1} (1-z)^{\frac{2\kappa}{\beta}(1-\theta(a-1))-1} (a-z)^{\frac{2\kappa}{\beta}-1} \left(\frac{1}{a-z} - \theta \right) dz \\ &= c_\theta \int_0^1 z^{u-1} (1-z)^{v-1} (a-z) \left(\frac{1}{a-z} - \theta \right) dz = 0. \end{aligned} \quad (152)$$

Now, consider the case $\alpha > \beta$. Let $\hat{\delta} := \alpha - \beta$, $\delta := \frac{2\hat{\delta}}{\beta}$, and $z^* := \sup\{z \in (0, 1) : \frac{1}{a-z} - \theta < 0\}$. Note that $\hat{\delta}, \delta > 0$ and $z^* = a - \frac{1}{\theta} \in (0, 1)$. Also note that for all $z \in (0, z^*)$ we have $\frac{1}{a-z} - \theta < 0$ and $(a-z)^\delta > (a-z^*)^\delta$. Furthermore, for all $z \in (z^*, 1)$ we have $\frac{1}{a-z} - \theta > 0$ and $(a-z)^\delta < (a-z^*)^\delta$. Together with (151) we thereby

obtain

$$\begin{aligned}
\int_0^1 \frac{1}{a-z} \Psi_\theta(dz) - \theta &= \int_0^1 \left(\frac{1}{a-z} - \theta \right) \Psi_\theta(dz) = \int_0^1 c_\theta z^{u-1} (1-z)^{v-1} (a-z)^{\frac{2\alpha}{\beta}-1} \left(\frac{1}{a-z} - \theta \right) dz \\
&= \int_0^{z^*} c_\theta z^{u-1} (1-z)^{v-1} (a-z)^{1+\delta} \left(\frac{1}{a-z} - \theta \right) dz + \int_{z^*}^1 c_\theta z^{u-1} (1-z)^{v-1} (a-z)^{1+\delta} \left(\frac{1}{a-z} - \theta \right) dz \\
&< c_\theta (a-z^*)^\delta \left(\int_0^{z^*} z^{u-1} (1-z)^{v-1} (a-z) \left(\frac{1}{a-z} - \theta \right) dz + \int_{z^*}^1 z^{u-1} (1-z)^{v-1} (a-z) \left(\frac{1}{a-z} - \theta \right) dz \right) \\
&= c_\theta (a-z^*)^\delta \int_0^1 z^{u-1} (1-z)^{v-1} (a-z) \left(\frac{1}{a-z} - \theta \right) dz = 0.
\end{aligned} \tag{153}$$

The case $\alpha < \beta$ can be proved analogously and thereby, we omit it here. This finishes the proof. \square

4.3 Proof of Theorem 1.4

Proof of Theorem 1.4. Applying Itô's lemma, we get for all $t \in [0, \infty)$ that

$$\begin{aligned}
\frac{1}{a-Z_t} - \frac{1}{a-Z_0} &= \int_0^t \frac{1}{(a-Z_s)^2} \left(\kappa(a-Z_s) \left((a-Z_s) \mathbb{E} \left[\frac{1}{a-Z_s} \right] - 1 \right) - \alpha Z_s (1-Z_s) \right) \\
&\quad + \frac{1}{2} \frac{2(a-Z_s)}{(a-Z_s)^4} \beta (a-Z_s) Z_s (1-Z_s) ds + \int_0^t \frac{1}{(a-Z_s)^2} \sqrt{\beta(a-Z_s) Z_s (1-Z_s)} dW_s \\
&= \int_0^t \kappa \left(\mathbb{E} \left[\frac{1}{a-Z_s} \right] - \frac{1}{a-Z_s} \right) - \frac{\alpha Z_s (1-Z_s)}{(a-Z_s)^2} + \frac{\beta Z_s (1-Z_s)}{(a-Z_s)^2} ds + \int_0^t \frac{1}{(a-Z_s)^2} \sqrt{\beta(a-Z_s) Z_s (1-Z_s)} dW_s.
\end{aligned} \tag{154}$$

After taking expectations we can apply Fubini's theorem to obtain for all $t \in [0, \infty)$ that

$$\begin{aligned}
\mathbb{E} \left[\frac{1}{a-Z_t} \right] - \mathbb{E} \left[\frac{1}{a-Z_0} \right] &= \int_0^t \kappa \left(\mathbb{E} \left[\frac{1}{a-Z_s} \right] - \mathbb{E} \left[\frac{1}{a-Z_s} \right] \right) - \alpha \mathbb{E} \left[\frac{Z_s (1-Z_s)}{(a-Z_s)^2} \right] + \beta \mathbb{E} \left[\frac{Z_s (1-Z_s)}{(a-Z_s)^2} \right] ds \\
&= (\beta - \alpha) \int_0^t \mathbb{E} \left[\frac{Z_s (1-Z_s)}{(a-Z_s)^2} \right] ds.
\end{aligned} \tag{155}$$

Since for all $s \in [0, \infty)$ it holds that $\mathbb{E} \left[\frac{Z_s (1-Z_s)}{(a-Z_s)^2} \right] \geq 0$ we conclude that the function $[0, \infty) \ni t \mapsto \mathbb{E} \left[\frac{1}{a-Z_t} \right] \in \left[\frac{1}{a}, \frac{1}{a-1} \right]$ converges monotonically non-increasing as $t \rightarrow \infty$ if $\alpha > \beta$, monotonically non-decreasing if $\alpha < \beta$, or is constant if $\alpha = \beta$.

First, assume $\alpha > \beta$. From (145) we see that δ_1 is an invariant measure for Z . So if $\mathbb{P}[Z_0 = 1] = 1$, then for all $t \in [0, \infty)$ it holds that $\mathbb{P}[Z_t = 1] = 1$. Now let $\mathbb{P}[Z_0 = 1] < 1$, implying $\mathbb{E} \left[\frac{1}{a-Z_0} \right] \in \left[\frac{1}{a}, \frac{1}{a-1} \right)$. Define $\theta := \lim_{t \rightarrow \infty} \mathbb{E} \left[\frac{1}{a-Z_t} \right]$ and fix it for the rest of the paragraph. Note that due to the monotonicity stated above we have $\theta \in \left[\frac{1}{a}, \frac{1}{a-1} \right)$. Aiming at a contradiction, we assume that $\theta \in \left(\frac{1}{a}, \frac{1}{a-1} \right)$. Choose any $\varepsilon \in \left(0, \frac{1}{a-1} - \theta \right)$ and fix it for the rest of the proof. By definition of θ there exists an $s_\varepsilon \in (0, \infty)$, such that for all $t \in [s_\varepsilon, \infty)$ it holds that $\mathbb{E} \left[\frac{1}{a-Z_t} \right] < \theta + \varepsilon$. Let $\tilde{W}: [0, \infty) \times \Omega \rightarrow \mathbb{R}$ be a Brownian motion with continuous sample paths, let $\tilde{Z}: [0, \infty) \times \Omega \rightarrow [0, 1]$ and $\tilde{Z}^{\theta+\varepsilon}: [0, \infty) \times \Omega \rightarrow [0, 1]$ be adapted processes with continuous sample paths that satisfy for all $t \in [0, \infty)$ \mathbb{P} -a.s.

$$\begin{aligned}
\tilde{Z}_t &= \tilde{Z}_0 + \int_0^t \left(\kappa(a - \tilde{Z}_s) \left((a - \tilde{Z}_s) \mathbb{E} \left[\frac{1}{a-Z_s} \right] - 1 \right) - \alpha \tilde{Z}_s (1 - \tilde{Z}_s) \right) ds \\
&\quad + \int_0^t \sqrt{\beta(a - \tilde{Z}_s) \tilde{Z}_s (1 - \tilde{Z}_s)} d\tilde{W}_s, \\
\tilde{Z}_t^{\theta+\varepsilon} &= \tilde{Z}_0^{\theta+\varepsilon} + \int_0^t \left(\kappa(a - \tilde{Z}_s^{\theta+\varepsilon}) \left((a - \tilde{Z}_s^{\theta+\varepsilon}) (\theta + \varepsilon) - 1 \right) - \alpha \tilde{Z}_s^{\theta+\varepsilon} (1 - \tilde{Z}_s^{\theta+\varepsilon}) \right) ds \\
&\quad + \int_0^t \sqrt{\beta(a - \tilde{Z}_s^{\theta+\varepsilon}) \tilde{Z}_s^{\theta+\varepsilon} (1 - \tilde{Z}_s^{\theta+\varepsilon})} d\tilde{W}_s,
\end{aligned} \tag{156}$$

such that $\tilde{Z}_0^{\theta+\varepsilon} = \tilde{Z}_0$ and such that \tilde{Z}_0 and Z_{s_ε} are equal in distribution. Then for each $t \in [s_\varepsilon, \infty)$ we have that Z_t and $\tilde{Z}_{t-s_\varepsilon}$ are equal in distribution and the drift term of $\tilde{Z}_{t-s_\varepsilon}$ is lower than that of $\tilde{Z}_{t-s_\varepsilon}^{\theta+\varepsilon}$. Together with the fact that the mapping $[0, 1] \ni z \mapsto \frac{1}{a-z}$ is strictly monotonically increasing this implies for all $t \in [s_\varepsilon, \infty)$ that

$$\mathbb{E}\left[\frac{1}{a-Z_t}\right] = \mathbb{E}\left[\frac{1}{a-Z_{t-s_\varepsilon}}\right] \leq \mathbb{E}\left[\frac{1}{a-\tilde{Z}_{t-s_\varepsilon}^{\theta+\varepsilon}}\right]. \quad (157)$$

Recall from Section 4.2 that for any $\eta \in (\frac{1}{a}, \frac{1}{a-1})$ we have that Ψ_η is the unique equilibrium distribution of \tilde{Z}^η . Combining this with (157) we obtain (see, e.g., Theorem V.54.5 in Rogers and Williams [34])

$$\theta = \lim_{t \rightarrow \infty} \mathbb{E}\left[\frac{1}{a-Z_t}\right] \leq \lim_{t \rightarrow \infty} \mathbb{E}\left[\frac{1}{a-\tilde{Z}_{t-s_\varepsilon}^{\theta+\varepsilon}}\right] = \int_0^1 \frac{1}{a-z} \Psi_{\theta+\varepsilon}(dz). \quad (158)$$

The dominated convergence theorem yields that the mapping $(\frac{1}{a}, \frac{1}{a-1}) \ni \eta \mapsto \Psi_\eta$ is continuous with respect to the weak topology. Applying this, (158) together with the fact that $\varepsilon \in (0, \frac{1}{a-1} - \theta)$ was arbitrarily chosen, and Lemma 4.2, we obtain the contradiction

$$\theta \leq \lim_{\delta \rightarrow 0} \int_0^1 \frac{1}{a-z} \Psi_{\theta+\delta}(dz) = \int_0^1 \frac{1}{a-z} \Psi_\theta(dz) < \theta. \quad (159)$$

Hence, we have $\theta = \frac{1}{a}$, implying

$$0 \leq \lim_{t \rightarrow \infty} \mathbb{E}[Z_t] \leq \lim_{t \rightarrow \infty} a^2 \mathbb{E}\left[\frac{Z_t}{a(a-Z_t)}\right] = \lim_{t \rightarrow \infty} a^2 \mathbb{E}\left[\frac{1}{a-Z_t}\right] - a^2 \frac{1}{a} = 0. \quad (160)$$

The case $\alpha < \beta$ can be proved analogously and we omit it here.

Finally, assume $\alpha = \beta$, define $\theta := \mathbb{E}[\frac{1}{a-Z_0}]$, and fix it for the rest of the proof. We see from (155) that $\mathbb{E}[\frac{1}{a-Z_t}]$ is constant in $t \in [0, \infty)$. Thus, assuming that Z_0 and Z_0^θ are equal in distribution we see from (145) and (146) that for all $t \in [0, \infty)$ it holds that Z_t and Z_t^θ are equal in distribution. Recall from Section 4.2 that Ψ_θ is the unique equilibrium distribution of Z^θ . Consequently, Ψ_θ is the unique equilibrium distribution of Z . This finishes the proof of Theorem 1.4. \square

5 Invasion of an altruistic defense allele

5.1 Setting

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $\kappa, \alpha, \beta \in (0, \infty)$, $a \in (1, \infty)$, and let $W(i): [0, \infty) \times \Omega \rightarrow \mathbb{R}$, $i \in \mathbb{N}$, be independent Brownian motions with continuous sample paths. For all $D \in \mathbb{N}$ let $X^D: [0, \infty) \times \{1, \dots, D\} \times \Omega \rightarrow [0, 1]$ be an adapted process with continuous sample paths that for all $t \in [0, \infty)$ and all $i \in \{1, \dots, D\}$ \mathbb{P} -a.s. satisfies

$$\begin{aligned} X_t^D(i) &= X_0^D(i) + \int_0^t (a - X_s^D(i)) \left((a - X_s^D(i)) \frac{1}{D} \sum_{j=1}^D \frac{1}{a - X_s^D(j)} - 1 \right) - \alpha X_s^D(i) (1 - X_s^D(i)) ds \\ &\quad + \int_0^t \sqrt{\beta(a - X_s^D(i)) X_s^D(i) (1 - X_s^D(i))} dW_s(i). \end{aligned} \quad (161)$$

Let $\tilde{a}: [0, \infty) \rightarrow [0, \infty)$ be a function defined by

$$[0, \infty) \ni x \mapsto \tilde{a}(x) := \kappa a \frac{\min\{x, 1\}}{a - \min\{x, 1\}} + (x - 1)^+. \quad (162)$$

Then, assuming there is positive mass only in deme 1, the dynamics in deme 1 follows asymptotically the following process Y . Let $Y: [0, \infty) \times \Omega \rightarrow [0, 1]$ be an adapted process with continuous sample paths such that for all $t \in [0, \infty)$ it \mathbb{P} -a.s. holds that

$$Y_t = Y_0 - \int_0^t \frac{\kappa}{a} Y_s (a - Y_s) + \alpha Y_s (1 - Y_s) ds + \int_0^t \sqrt{\beta(a - Y_s) Y_s (1 - Y_s)} dW_s(1). \quad (163)$$

In addition, let Q_Y be the excursion measure which satisfies $Q_Y = \lim_{0 < \varepsilon \rightarrow 0} \frac{1}{\varepsilon} \mathbb{P}[Y \in \cdot | Y_0 = \varepsilon]$ in a suitable sense; see Pitman and Yor [29] and Hutzenthaler [15] for details. Asymptotically in the many-demes limit, every deme with population path $\chi \in C([0, \infty), [0, 1])$ populates demes through migration and these new populations are given by a Poisson point process with intensity measure $\tilde{a}(\chi_t) dt \times Q_Y(d\psi)$. Now let $(V_t)_{t \in [0, \infty)}$ be the total mass process of the associated tree of excursions with initial island measure that equals the distribution of Y in (163) and excursion measure Q_Y .

5.2 Survival or extinction of an invading altruistic defense allele

Proposition 5.1. *Assume the setting of Section 5.1. Let $x \in (0, 1]$ and assume $Y_0 = x = V_0$. Then the total mass process dies out (i.e., converges in probability to zero as $t \rightarrow \infty$) if and only if*

$$\alpha \geq \beta. \quad (164)$$

Proof. Define the functions $s: [0, 1] \rightarrow [0, \infty)$ and $S: [0, 1] \rightarrow [0, \infty)$ by $[0, 1] \ni z \mapsto s(z) := \exp\left(-\int_0^z \frac{-\frac{\kappa}{a}x(a-x) - \alpha x(1-x)}{\frac{1}{2}\beta(a-x)x(1-x)} dx\right)$ and $[0, 1] \ni y \mapsto S(y) := \int_0^y s(z) dz$. Note that for all $z \in [0, 1]$ it holds that

$$s(z) = \exp\left(\int_0^z \frac{2\kappa}{a\beta} \frac{1}{1-x} + \frac{2\alpha}{\beta} \frac{1}{a-x} dx\right) = (1-z)^{\frac{-2\kappa}{a\beta}} \left(\frac{a-z}{a}\right)^{\frac{-2\alpha}{\beta}} \quad (165)$$

and

$$S(z) = \int_0^z s(x) dx \leq zs(z). \quad (166)$$

We will apply Theorem 5 from Hutzenthaler [15] to show the result. First, we verify that the assumptions of the aforementioned theorem are satisfied. Using (166), we see that

$$\int_0^{\frac{1}{2}} S(y) \frac{2}{\beta(a-y)y(1-y)s(y)} dy \leq \int_0^{\frac{1}{2}} \frac{2}{\beta(a-y)(1-y)} dy \leq \frac{1}{2} \frac{2}{\beta(a-\frac{1}{2})(1-\frac{1}{2})} < \infty. \quad (167)$$

Furthermore, we get

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^{\frac{1}{2}} \frac{-\frac{\kappa}{a}(a-y)y - \alpha y(1-y)}{\frac{1}{2}\beta(a-y)y(1-y)} dy &= \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^{\frac{1}{2}} \frac{-2\kappa}{a\beta(1-y)} - \frac{2\alpha}{\beta(a-y)} dy \\ &= \lim_{\varepsilon \rightarrow 0} \left(\frac{2\kappa}{a\beta} (\ln(1 - \frac{1}{2}) - \ln(1 - \varepsilon)) + \frac{2\alpha}{\beta} (\ln(a - \frac{1}{2}) - \ln(a - \varepsilon)) \right) \\ &= \frac{2\kappa}{a\beta} \ln(1 - \frac{1}{2}) + \frac{2\alpha}{\beta} (\ln(a - \frac{1}{2}) - \ln(a)) \in (-\infty, \infty). \end{aligned} \quad (168)$$

From (165) as well as the fact that $\frac{2\kappa}{a\beta} > 0$ we see that

$$\begin{aligned} \int_{\frac{1}{2}}^1 \frac{\tilde{a}(y)}{\frac{1}{2}\beta(a-y)y(1-y)s(y)} dy &= \int_{\frac{1}{2}}^1 \frac{\kappa a - y}{\frac{1}{2}\beta(a-y)y(1-y)} (1-y)^{\frac{2\kappa}{a\beta}} \left(\frac{a-y}{a}\right)^{\frac{2\alpha}{\beta}} dy \\ &= 2 \frac{\kappa a}{\beta} a^{-\frac{2\alpha}{\beta}} \int_{\frac{1}{2}}^1 (1-y)^{\frac{2\kappa}{a\beta}-1} (a-y)^{\frac{2\alpha}{\beta}-2} dy \\ &\leq 2 \frac{\kappa a}{\beta} a^{-\frac{2\alpha}{\beta}} \left((a - \frac{1}{2})^{\frac{2\alpha}{\beta}-2} + (a-1)^{\frac{2\alpha}{\beta}-2} \right) \int_{\frac{1}{2}}^1 (1-y)^{\frac{2\kappa}{a\beta}-1} dy < \infty. \end{aligned} \quad (169)$$

We obtain from (167), (168), and (169) together with a straightforward adaptation of Lemmas 9.6, 9.9, and 9.10 in Hutzenthaler [15] to the state space $[0, 1]$ that the assumptions of Theorem 5 in Hutzenthaler [15] are satisfied. Applying the aforementioned theorem shows that the total mass process dies out if and only if

$$\int \int_0^\infty \tilde{a}(\chi_t) dt Q_Y(d\chi) \leq 1. \quad (170)$$

Moreover, a straight forward adaptation of Lemma 9.8 in Hutzenthaler [15] to the state space $[0, 1]$ together with (165) shows that

$$\int \int_0^\infty \tilde{a}(\chi_t) dt Q_Y(d\chi) = \int_0^1 \frac{\kappa a \frac{y}{a-y}}{\frac{1}{2}\beta(a-y)y(1-y)} (1-y)^{\frac{2\kappa}{a\beta}} \left(\frac{a-y}{a}\right)^{\frac{2\alpha}{\beta}} dy. \quad (171)$$

Observe that we have $\frac{2\kappa}{a\beta} \int_0^1 (1-y)^{\frac{2\kappa}{a\beta}-1} dy = 1$. Combining this with (170) and (171) we see that the total mass process dies out if and only if

$$\begin{aligned} 0 &\geq \int_0^1 \frac{\kappa a \frac{y}{a-y}}{\frac{1}{2}\beta(a-y)y(1-y)} (1-y)^{\frac{2\kappa}{a\beta}} \left(\frac{a-y}{a}\right)^{\frac{2\alpha}{\beta}} dy - 1 \\ &= \frac{2\kappa}{a\beta} \int_0^1 (1-y)^{\frac{2\kappa}{a\beta}-1} \left(\frac{a-y}{a}\right)^{\frac{2\alpha}{\beta}-2} dy - 1 = \frac{2\kappa}{a\beta} \int_0^1 (1-y)^{\frac{2\kappa}{a\beta}-1} \left(\left(\frac{a-y}{a}\right)^{\frac{2\alpha}{\beta}-2} - 1 \right) dy. \end{aligned} \quad (172)$$

Consequently, the total mass process dies out if and only if $\alpha \geq \beta$. This finishes the proof of Proposition 5.1. \square

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